28. THE PLANE

PLANE *A plane is a surface such that a line segment joining any two points on it lies wholly on it.*

NORMAL TO A PLANE *A straight line which is perpendicular to every line lying on a plane is called a normal to the plane.*

All the normals to a plane are parallel to each other.

General Equation of a Plane in the Cartesian Form

THEOREM 1 *Every equation $ax + by + cz + d = 0$ of the first degree in x, y, z always represents a plane. Also, a, b, c are the direction ratios of the normal to this plane.*

PROOF Let us consider a surface represented by the equation

$$ax + by + cz + d = 0. \qquad \ldots \text{(i)}$$

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the surface represented by (i). Then,

$$ax_1 + by_1 + cz_1 + d = 0 \qquad \ldots \text{(ii)}$$

and, $ax_2 + by_2 + cz_2 + d = 0.$ $\ldots$ (iii)

Multiplying (iii) by λ and adding it to (ii), we get

$$a(\lambda x_2 + x_1) + b(\lambda y_2 + y_1) + c(\lambda z_2 + z_1) + d(\lambda + 1) = 0$$

$$\Rightarrow \quad a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c\left(\frac{\lambda z_2 + z_1}{\lambda + 1}\right) + d = 0$$

$$\Rightarrow \quad \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right) \text{ lies on surface (i), when } \lambda \neq -1.$$

But, these are the general coordinates of a point which divides AB in the ratio $\lambda : 1$.

Since λ may take any real value other than –1, it follows that every point of AB lies on (i).

Hence, $ax + by + cz + d = 0$ represents a plane.

To Show that a, b, c are the Direction Ratios of the Normal to the Plane

Subtracting (ii) from (iii), we get

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

$\Rightarrow$ a line with direction ratios a, b, c is perpendicular to an arbitrary line AB taken on plane (i) [$\because$ $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$ are d.r.'s of AB]

$\Rightarrow$ a line with d.r.'s a, b, c is perpendicular to the plane (i)

$\Rightarrow$ a, b, c are the direction ratios of the normal to the plane (i).

Hence, $ax + by + cz + d = 0$ represents a plane, and a, b, c are the direction ratios of the normal to this plane.

Equation of a Plane Passing through a Given Point

THEOREM 2 *The equation of a plane passing through a point* $P(x_1, y_1, z_1)$ *is* $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, *where* a, b, c *are constants.*

PROOF The general equation of a plane is

$$ax + by + cz + d = 0. \quad \ldots \text{(i)}$$

If this plane passes through the point $P(x_1, y_1, z_1)$ then

$$ax_1 + by_1 + cz_1 + d = 0. \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the general equation of a plane passing through the point $P(x_1, y_1, z_1)$.

Equation of a Plane in the Intercept Form

THEOREM 3 *If a plane makes intercepts of lengths* a, b, c *with the x-axis, y-axis and z-axis respectively, the equation of the plane is* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

[CBSE 2004C]

PROOF Let O be the origin, and let the plane meet the coordinate axes at A, B, C respectively such that

$$OA = a, OB = b \text{ and } OC = c.$$

So, the coordinates of these points are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let the equation of the given plane be

$$Ax + By + Cz + D = 0. \quad \ldots \text{(i)}$$

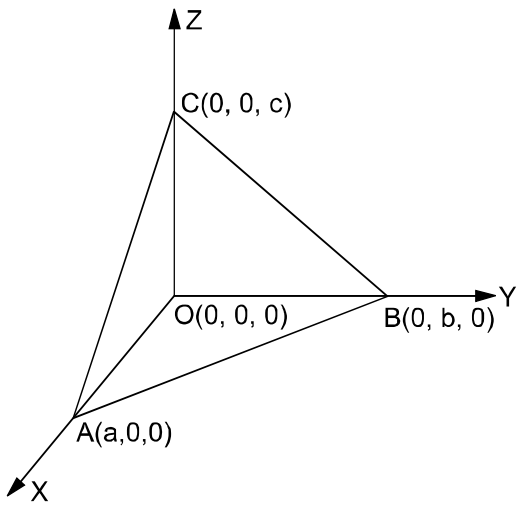
Since the given plane does not pass through $O(0, 0, 0)$, $D \neq 0$.

Also, since (i) passes through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$, we have

$$Aa + D = 0 \Rightarrow A = \frac{-D}{a},$$

$$Bb + D = 0 \Rightarrow B = \frac{-D}{b},$$

$$Cc + D = 0 \Rightarrow C = \frac{-D}{c}.$$



Putting these values in (i), we get

$$\frac{-Dx}{a} - \frac{Dy}{b} - \frac{Dz}{c} + D = 0$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{[on dividing throughout by } -D\text{]}.$$

This is the required equation of the plane in the intercept form.

Equation of a Plane Passing through Three Given Points

Suppose that a plane passes through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$and $C(x_3, y_3, z_3)$.

We know that the equation of a plane passing through the point $A(x_1, y_1, z_1)$ is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \qquad \text{... (i)}$$

If this plane passes through the points $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, we have

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \qquad \text{... (ii)}$$

and $$a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0. \qquad \text{... (iii)}$$

On eliminating a, b, c from (i), (ii) and (iii), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

This is the required equation of the plane passing through three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and (x_3, y_3, z_3).

SUMMARY OF THE RESULTS

1. The general equation of a plane is $ax + by + cz + d = 0$. The d.r.'s of the normal to the plane are a, b, c.
2. The equation of the plane passing through the point $P(x_1, y_1, z_1)$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.
3. If a plane makes intercepts a, b and c with the x-axis, y-axis and z-axis respectively, then its equation is
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$
4. The equation of a plane passing through three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of the plane passing through the points* $A(2, 5, -3)$, $B(-2, -3, 5)$ *and* $C(5, 3, -3)$.

SOLUTION The equation of the plane passing through the point $A(2, 5, -3)$ is given by

$$a(x-2) + b(y-5) + c(z+3) = 0. \quad \text{... (i)}$$

If this plane passes through the points $B(-2, -3, 5)$ and $C(5, 3, -3)$, then we have

$$a(-2-2) + b(-3-5) + c(5+3) = 0 \Rightarrow -4a - 8b + 8c = 0 \quad \text{... (ii)}$$

$$a(5-2) + b(3-5) + c(-3+3) = 0 \Rightarrow 3a - 2b + 0c = 0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(0+16)} = \frac{b}{(24-0)} = \frac{c}{(8+24)} \Rightarrow \frac{a}{16} = \frac{b}{24} = \frac{c}{32}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{4} = k \text{ (say)} \Rightarrow a = 2k,\ b = 3k \text{ and } c = 4k.$$

Putting these values of a, b and c in (i), we get

$$2k(x-2) + 3k(y-5) + 4k(z+3) = 0$$

$$\Rightarrow 2(x-2) + 3(y-5) + 4(z+3) = 0$$

$$\Rightarrow 2x + 3y + 4z - 7 = 0.$$

Hence, the required equation of the plane is $2x + 3y + 4z - 7 = 0$.

Alternate method

We know that equation of the plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

Here,

$$(x_1 = 2, y_1 = 5, z_1 = -3), (x_2 = -2, y_2 = -3, z_2 = 5) \text{ and}$$
$$(x_3 = 5, y_3 = 3, z_3 = -3).$$

$\therefore$ the required equation of the plane is

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0.$$

$$\Rightarrow (x-2)(0+16) - (y-5)(0-24) + (z+3)(8+24) = 0$$

$$\Rightarrow 16(x-2) + 24(y-5) + 32(z+3) = 0 \Rightarrow 16x + 24y + 32z - 56 = 0$$

$$\Rightarrow 2x + 3y + 4z - 7 = 0.$$

Hence, the required equation of the plane is $2x + 3y + 4z - 7 = 0$.

EXAMPLE 2 *Show that the four points* $A(1, -1, 1)$, $B(2, 3, 1)$, $C(1, 2, 3)$ *and* $D(0, -2, 3)$ *are coplanar. Find the equation of the plane containing them.*

SOLUTION We know that the equation of the plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

Here,

$(x_1 = 1, y_1 = -1, z_1 = 1)$, $(x_2 = 2, y_2 = 3, z_2 = 1)$ and $(x_3 = 1, y_3 = 2, z_3 = 3)$.

$\therefore$ the equation of the plane passing through the points A, B and C is given by

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2-1 & 3+1 & 1-1 \\ 1-1 & 2+1 & 3-1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y+1 & z-1 \\ 1 & 4 & 0 \\ 0 & 3 & 2 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(8-0) - (y+1)(2-0) + (z-1)(3-0) = 0$$

$$\Rightarrow 8(x-1) - 2(y+1) + 3(z-1) = 0 \Rightarrow 8x - 2y + 3z = 13. \quad \text{... (i)}$$

Putting $x = 0$, $y = -2$ and $z = 3$ in (i), we get

$$\text{LHS} = (8 \times 0) - 2 \times (-2) + 3 \times 3 = (0 + 4 + 9) = 13 = \text{RHS}.$$

Thus, the point $D(0, -2, 3)$ lies on the plane (i).

Hence, the given four points are coplanar and the equation of the plane containing them is $8x = 2y + 3z = 13$.

EXAMPLE 3 *Find the equation of the plane which cuts off intercepts* $3, 6$ *and* -4 *from the axes of coordinates.*

SOLUTION We know that the equation of a plane which cuts off intercepts a, b, c from the x-axis, y-axis and z-axis respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Here $a = 3$, $b = 6$ and $c = -4$.

Hence, the required equation of the plane is

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{-4} = 1 \Rightarrow 4x + 2y - 3z = 12.$$

EXAMPLE 4 *Reduce the equation of the plane* $2x - 3y + z = 6$ *to intercept form and find its intercepts on the coordinate axes.*

SOLUTION We have

$$2x - 3y + z = 6 \Rightarrow \frac{2x}{6} - \frac{3y}{6} + \frac{z}{6} = 1 \Rightarrow \frac{x}{3} + \frac{y}{-2} + \frac{z}{6} = 1.$$

Hence, the equation of the given plane in intercept form is $\frac{x}{3} + \frac{y}{-2} + \frac{z}{6} = 1$.

The intercepts of the given plane on x-axis, y-axis and z-axis are 3, –2 and 6 respectively.

EXAMPLE 5 *A plane meets the coordinate axes in A, B, C such that the centroid of* $\triangle ABC$ *is* (p, q, r). *Show that the equation of the plane is* $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

SOLUTION Let the required equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. ... (i)

Then, clearly the plane meets the coordinate axis at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Since the centroid of $\triangle ABC$ is $G(p, q, r)$, we have

$$\frac{a+0+0}{3}=p,\ \frac{0+b+0}{3}=q \text{ and } \frac{0+0+c}{3}=r$$

$\Rightarrow$ $a = 3p$, $b = 3q$ and $c = 3r$.

Putting these values of a, b, c in (i) we get

$$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3.$$

Hence, the required equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

EXAMPLE 6 *A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is constant. Show that the plane passes through a fixed point.*

SOLUTION Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \quad \ldots \text{(i)}$$

Then, it makes intercepts a, b, c with the coordinate axes.

$\therefore$ $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=k$, where k is a constant (given)

$$\Rightarrow \frac{1}{ka}+\frac{1}{kb}+\frac{1}{kc}=1 \Rightarrow \frac{1}{a}\left(\frac{1}{k}\right)+\frac{1}{b}\left(\frac{1}{k}\right)+\frac{1}{c}\left(\frac{1}{k}\right)=1$$

$\Rightarrow$ $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$ satisfies (i).

Hence, the given plane passes through a fixed point $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$.

EXERCISE 28A

1. Find the equation of the plane passing through each group of points:
   (i) $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$
   (ii) $A(0, -1, -1)$, $B(4, 5, 1)$ and $C(3, 9, 4)$ **[CBSE 2006]**
   (iii) $A(-2, 6, -6)$, $B(-3, 10, -9)$ and $C(-5, 0, -6)$.
2. Show that the four points $A(3, 2, -5)$, $B(-1, 4, -3)$, $C(-3, 8, -5)$ and $D(-3, 2, 1)$ are coplanar. Find the equation of the plane containing them.

3. Show that the four points $A(0, -1, 0)$, $B(2, 1, -1)$, $C(1, 1, 1)$ and $D(3, 3, 0)$ are coplanar. Find the equation of the plane containing them.
4. Write the equation of the plane whose intercepts on the coordinate axes are 2, – 4 and 5 respectively.
5. Reduce the equation of the plane $4x - 3y + 2z = 12$ to the intercept form, and hence find the intercepts made by the plane with the coordinate axes.
6. Find the equation of the plane which passes through the point (2, –3, 7) and makes equal intercepts on the coordinate axes.
7. A plane meets the coordinate axes at A, B and C respectively such that the centroid of $\triangle ABC$ is (1, –2, 3). Find the equation of the plane.
8. Find the Cartesian and vector equations of a plane passing through the point (1, 2, 3) and perpendicular to a line with direction ratios 2, 3, –4.

[CBSE 2005C]

9. If O be the origin and $P(1, 2, -3)$ be a given point, then find the equation of the plane passing through P and perpendicular to OP.

ANSWERS (EXERCISE 28A)

1. (i) $5x + 2y - 3z = 17$ (ii) $5x - 7y + 11z + 4 = 0$ (iii) $2x - y - 2z = 2$

2. $x + y + z = 0$ **3.** $4x - 3y + 2z = 3$ **4.** $10x - 5y + 4z = 20$

5. $\frac{x}{3} + \frac{y}{-4} + \frac{z}{6} = 1$, Intercepts on the axes are 3, –4, 6.

6. $x + y + z = 6$ **7.** $\frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$

8. $2x + 3y - 4z + 4 = 0$, $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0$

9. $x + 2y - 3z - 14 = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28A)

5. $4x - 3y + 2z = 12 \Rightarrow \frac{x}{3} + \frac{y}{-4} + \frac{z}{6} = 1$ [on dividing each term by 12].

6. Let it make intercept a on each of the coordinate axes.

Then, its equation is $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1 \Rightarrow x + y + z = a$.

Putting $x = 2, y = -3, z = 7$, we get $a = 2 + (-3) + 7 = 6$.

So, the required equation of the plane is $x + y + z = 6$.

7. Let the plane meet the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Then, $\frac{a}{3} = 1, \frac{b}{3} = -2, \frac{c}{3} = 3 \Rightarrow a = 3, b = -6, c = 9$.

$\therefore$ required equation of the plane is $\frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$.

8. Required equation is $2(x - 1) + 3(y - 2) - 4(z - 3) = 0 \Rightarrow 2x + 3y - 4z + 4 = 0$.

Its vector equation is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0, \text{ i.e., } \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0.$$

9. Let the required equation of the plane passing through the point $A(1, 2, -3)$ be

$$a(x-1) + b(y-2) + c(z+3) = 0 \qquad \ldots \text{(i)}$$

D.r.'s of OP are $(1-0), (2-0), (-3-0)$, i.e., $1, 2, -3$.

Since the plane is perpendicular to OP, so the normal to the plane is parallel to OP.

$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{-3} = k$ (say) $\Rightarrow \; a = k,\ b = 2k$ and $c = -3k$.

$\therefore$ required equation of the plane is $k(x-1) + 2k(y-2) - 3k(z+3) = 0$

$\Rightarrow \quad (x-1) + 2(y-2) - 3(z+3) = 0.$

$\Rightarrow \quad (x + 2y - 3z) + (-1 - 4 - 9) = 0$

$\Rightarrow \quad x + 2y - 3z - 14 = 0.$

EQUATIONS OF A PLANE IN VARIOUS FORMS

Equation of a Plane in the Normal Form

Vector Form

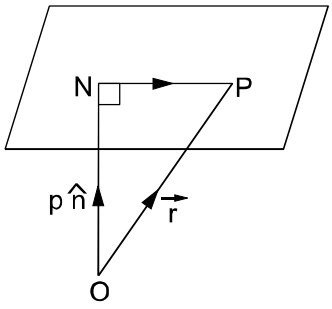
THEOREM 1 *If $\hat{n}$ is a unit vector normal to a given plane, directed from the origin to the plane, and p is the length of the perpendicular drawn from the origin to the plane then the vector equation of the plane is $\vec{r} \cdot \hat{n} = p$.*

PROOF Let O be the origin, and let ON be the perpendicular drawn from O to the given plane. Let $ON = p$.

Let $\hat{n}$ be a unit vector along $\overrightarrow{ON}$.

Then, $\overrightarrow{ON} = p\hat{n}$.

Let P be an arbitrary point on the plane, and let the position vector of P be $\vec{r}$.



Then, $\overrightarrow{OP} = \vec{r}$.

Since $\overrightarrow{NP}$ lies on the plane, $\overrightarrow{NP}$ is perpendicular to $\hat{n}$.

$\therefore \quad \overrightarrow{NP} \cdot \hat{n} = 0$

$\Rightarrow \quad (\overrightarrow{OP} - \overrightarrow{ON}) \cdot \hat{n} = 0 \Rightarrow (\vec{r} - p\hat{n}) \cdot \hat{n} = 0$

$\Rightarrow \quad \vec{r} \cdot \hat{n} - p\,\hat{n} \cdot \hat{n} = 0 \Rightarrow \vec{r} \cdot \hat{n} = p.$

Hence, the required equation of the plane is $\vec{r} \cdot \vec{n} = p$.

REMARK The equation of a plane which is at a distance p from the origin and which is perpendicular to $\hat{n}$ is $\vec{r} \cdot \hat{n} = p$.

COROLLARY *If $\vec{n}$ is a vector normal to a given plane then $\vec{r} \cdot \hat{n} = q$ represents a plane.*

PROOF $\vec{r} \cdot \vec{n} = q \Rightarrow \vec{r} \cdot \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{q}{|\vec{n}|} \Rightarrow \vec{r} \cdot \hat{n} = p$, where $\dfrac{q}{|\vec{n}|} = p$.

But, $\vec{r} \cdot \hat{n} = p$ represents a plane.

Hence, $\vec{r} \cdot \vec{n} = q$ also represents a plane.

Cartesian Forms

(i) Normal Form:

Let l, m, n be the d.c.'s of $\hat{n}$ and $P(x, y, z)$ be the given point.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$.

On substituting these values in $\vec{r} \cdot \hat{n} = p$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p$$

$$\Rightarrow \quad lx + my + nz = p.$$

This is known as the *normal form* of the Cartesian equation of a plane.

(ii) General Form:

Let a, b, c be the d.r.'s of $\vec{n}$ and $P(x, y, z)$ be the given point.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$.

On substituting these values $\vec{r} \cdot \vec{n} = q$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = q$$

$$\Rightarrow \quad ax + by + cz = q \quad \Rightarrow \quad ax + by + cz + d = 0, \text{ where } d = -q.$$

The equation $ax + by + cz + d = 0$ is called the *General Form* of the Cartesian equation of a plane.

Equation of a Plane Passing through a Given Point and Perpendicular to a Given Vector

Vector Form

THEOREM 1 *The vector equation of a plane passing through a point A with position vector $\vec{a}$ and perpendicular to a given vector $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.*

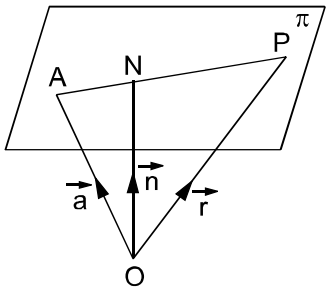
PROOF Let O be the origin, and let π be the given plane.

Let A be a given point on the plane with position vector $\vec{a}$.

Let P be an arbitrary point on the plane with position vector $\vec{r}$.

Then, $\overrightarrow{OA} = \vec{a}$, and $\overrightarrow{OP} = \vec{r}$.

$\therefore \quad \overrightarrow{AP} = (\overrightarrow{OP} - \overrightarrow{OA}) = (\vec{r} - \vec{a})$.



Let $\overrightarrow{ON} = \vec{n}$ be normal to the plane.

Now, $\overrightarrow{AP}$ lies in the plane and $\vec{n}$ is normal to the plane.

$\therefore \quad \overrightarrow{AP} \perp \vec{n} \Rightarrow \overrightarrow{AP} \cdot \vec{n} = 0 \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0.$

Hence, the equation of the plane passing through the point with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

COROLLARY *The vector equation of the plane passing through the origin and perpendicular to $\vec{n}$ is $\vec{r} \cdot \vec{n} = 0$.*

Cartesian Form

THEOREM 2 *The Cartesian equation of a plane passing through a point $A(x_1, y_1, z_1)$ and perpendicular to a line having d.r.'s a, b, c is*

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

PROOF Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$.

Substituting these values in $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, we get

$$\{(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\} \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$$

$$\Rightarrow \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is known as the equation of a plane in ***one-point form.***

SUMMARY

1. (i) Let a plane be at a distance p form the origin and let $\hat{n}$ be a unit vector perpendicular to the plane.
   Then, equation of the plane is $\vec{r} \cdot \hat{n} = p$.
   (ii) The Cartesian form of equation of this plane is $lx + my + nz = p$, where l, m, n are the d.c.'s of normal to the plane.
   (iii) The general equation of a plane is $\vec{r} \cdot \vec{n} = q$.
   (iv) Its Cartesian form is $ax + by + cz + d = 0$, where a, b, c are the d.r.'s of $\vec{n}$.
2. (i) The vector equation of a plane passing through a point A with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.
   (ii) The Cartesian equation of a plane passing through a point $A(x_1, y_1, z_1)$ and perpendicular to a line having d.r.'s a, b, c is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane which is at a distance of* 5 *units from the origin and which has* $\hat{j}$ *as the unit vector normal to it.*

SOLUTION Clearly, the required equation of the plane is $\vec{r} \cdot \hat{j} = 5$.

EXAMPLE 2 *Find the vector equation of a plane which is at a distance of* 6 *units from the origin and which is normal to the vector* $(\hat{i} + 2\hat{j} - 2\hat{k})$.

SOLUTION Here $\vec{n} = (\hat{i} + 2\hat{j} - 2\hat{k})$ and $p = 6$.

$$\therefore \quad \hat{n} = \frac{(\hat{i} + 2\hat{j} - 2\hat{k})}{\sqrt{1^2 + 2^2 + (-2)^2}} = \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right).$$

So, the vector equation of the plane is

$$\vec{r} \cdot \hat{n} = p \Rightarrow \vec{r} \cdot \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = 6$$

$$\Rightarrow \vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 18.$$

Hence, the required vector equation of the plane is

$$\vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 18.$$

EXAMPLE 3 *Find the vector equation of a plane passing through a point having position vector* $(2\hat{i} - \hat{j} + \hat{k})$ *and perpendicular to the vector* $(4\hat{i} + 2\hat{j} - 3\hat{k})$. *Also reduce it to Cartesian form.*

SOLUTION Here $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$ and $\vec{n} = (4\hat{i} + 2\hat{j} - 3\hat{k})$.

Clearly, the required vector equation is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = (2\hat{i} - \hat{j} + \hat{k}) \cdot (4\hat{i} + 2\hat{j} - 3\hat{k})$$

$$= (8 - 2 - 3) = 3.$$

Hence, the vector equation of the given plane is

$$\vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3 \qquad \ldots (i)$$

Reduction to Cartesian Form:

Putting $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3 \Rightarrow 4x + 2y - 3z = 3.$$

Hence, the equation of the given plane in Cartesian form is $4x + 2y - 3z = 3$.

EXAMPLE 4 *Find a unit vector normal to the plane* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$.

SOLUTION The equation of the given plane is

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})+14=0$$

$$\Leftrightarrow \quad \vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=-14 \quad \Rightarrow \quad \vec{r}\cdot(-2\hat{i}+3\hat{j}-6\hat{k})=14$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n}=14, \text{ where } \vec{n}=(-2\hat{i}+3\hat{j}-6\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{14}{|\vec{n}|}, \text{ where } |\vec{n}|=\sqrt{(-2)^2+3^2+(-6)^2}=7$$

$$\Rightarrow \quad \vec{r}\cdot\frac{(-2\hat{i}+3\hat{j}-6\hat{k})}{7}=\frac{14}{7}\Rightarrow \vec{r}\cdot\left(-\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=2.$$

Hence, the unit vector normal to the given plane is

$$\hat{n}=\left(-\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right).$$

EXAMPLE 5 *Find the direction cosines of the perpendicular from the origin to the plane* $\vec{r}\cdot(6\hat{i}-3\hat{j}-2\hat{k})+3=0.$

SOLUTION Clearly, we have to find the direction cosines of the normal to the given plane.

The given equation may be written as

$$\vec{r}\cdot(6\hat{i}-3\hat{j}-2\hat{k})=-3\Rightarrow \vec{r}\cdot(-6\hat{i}+3\hat{j}+2\hat{k})=3$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n}=3, \text{ where } \vec{n}=(-6\hat{i}+3\hat{j}+2\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{3}{|\vec{n}|}, \text{ where } |\vec{n}|=\sqrt{(-6)^2+3^2+2^2}=7$$

$$\Rightarrow \quad \vec{r}\cdot\frac{(-6\hat{i}+3\hat{j}+2\hat{k})}{7}=\frac{3}{7}\Rightarrow \vec{r}\cdot\left(-\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}\right)=\frac{3}{7}.$$

Hence, the direction cosines of the normal to the given plane are

$$\left(-\frac{6}{7},\frac{3}{7},\frac{2}{7}\right).$$

EXAMPLE 6 *Find the Cartesian equation of a plane whose vector equation is* $\vec{r}\cdot(2\hat{i}+5\hat{j}-4\hat{k})=3.$

SOLUTION We have

$$\vec{r}\cdot(2\hat{i}+5\hat{j}-4\hat{k})=3\Leftrightarrow(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+5\hat{j}-4\hat{k})=3$$

$$\Leftrightarrow \quad 2x+5y-4z=3.$$

Hence, the required equation is $2x+5y-4z=3$.

EXAMPLE 7 *Find the vectors equation of a plane whose Cartesian equation is* $2x - 3y + 4z + 6 = 0$.
Find the direction cosines of the normal to the plane and its distance from the origin.

SOLUTION The given equation of the plane is

$$2x - 3y + 4z = -6 \Rightarrow -2x + 3y - 4z = 6. \quad \text{... (i)}$$

If $\vec{r}$ is the position vector of any point $P(x, y, z)$ on plane (i), then we may write it as

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-2\hat{i} + 3\hat{j} - 4\hat{k}) = 6$$

$$\Rightarrow \quad \vec{r} \cdot (-2\hat{i} + 3\hat{j} - 4\hat{k}) = 6$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = 6, \text{ where } \vec{n} = (-2\hat{i} + 3\hat{j} - 4\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{6}{|\vec{n}|}, \text{ where } |\vec{n}| = \sqrt{(-2)^2 + 3^2 + (-4)^2} = \sqrt{29}$$

$$\Rightarrow \quad \vec{r} \cdot \left(\frac{-2}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{4}{\sqrt{29}}\hat{k} \right) = \frac{6}{\sqrt{29}}.$$

$\therefore$ d.c.'s of normal to the given plane are $\frac{-2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-4}{\sqrt{29}}$, and its distance from the origin is $\frac{6}{\sqrt{29}}$.

EXAMPLE 8 *Find the vector equation of the plane whose Cartesian equation is* $5y + 8 = 0$.
Find the direction cosines of the normal to the plane and its distance from the origin.

SOLUTION The given equation of the plane is

$$5y = -8 \Rightarrow -5y = 8 \Rightarrow 0 \cdot x - 5y + 0z = 8. \quad \text{... (i)}$$

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on plane (i). Then, the above equation may be written as

$$\vec{r} \cdot (0\hat{i} - 5\hat{j} + 0\hat{k}) = 8 \Rightarrow \vec{r} \cdot \vec{n} = 8, \text{ where } \vec{n} = 0\hat{i} - 5\hat{j} + 0\hat{k}$$

$$\Rightarrow \quad \vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{8}{|\vec{n}|}, \text{ where } |\vec{n}| = \sqrt{0^2 + (-5)^2 + 0^2} = 5$$

$$\Rightarrow \quad \vec{r} \cdot \left(\frac{0}{5}\hat{i} - \frac{5}{5}\hat{j} + \frac{0}{5}\hat{k} \right) = \frac{8}{5} \Rightarrow \vec{r} \cdot (0\hat{i} - \hat{j} + 0\hat{k}) = \frac{8}{5}.$$

$\therefore$ d.c.'s of the normal to the plane are 0, –1, 0 and its distance from the origin is $\frac{8}{5}$ units.

EXAMPLE 9 *Find the Cartesian form of the equation of the plane*

$$\vec{r} = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}.$$

SOLUTION We have

$$\vec{r} = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}$$

$$\Leftrightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}$$

$$\Leftrightarrow \quad x = s - 2t,\ y = 3 - t \text{ and } z = 2s + t$$

$$\Leftrightarrow \quad x - 2y = (s - 6) \text{ and } y + z = (3 + 2s) \qquad [\text{eliminating } t]$$

$$\Leftrightarrow \quad x - 2y + 6 = \frac{1}{2}(y + z - 3) \qquad [\text{equating the values of } s]$$

$$\Leftrightarrow \quad 2x - 4y + 12 = y + z - 3 \quad \Leftrightarrow \quad 2x - 5y - z + 15 = 0.$$

This is the required Cartesian form of the equation of the given plane.

EXAMPLE 10 *Find the vector and Cartesian equations of the plane which passes through the point* $(5, 2, -4)$ *and perpendicular to the line with direction ratios* $2, 3, -1$.

SOLUTION The plane passes through the point $A(5, 2, -4)$ whose position vector is $\vec{a} = (5\hat{i} + 2\hat{j} - 4\hat{k})$ and the normal vector $\vec{n}$ perpendicular to the plane is $\vec{n} = (2\hat{i} + 3\hat{j} - \hat{k})$.

So, the vector equation of the plane is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = (5\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k})$$

$$= (10 + 6 + 4) = 20.$$

Hence, the required vector equation of the plane is

$$\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) = 20. \qquad \ldots (i)$$

Cartesian Form:

Taking $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, equation (i) may be written as

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 20 \Rightarrow 2x + 3y - z = 20.$$

Hence, the required Cartesian equation is $2x + 3y - z = 20$.

EXAMPLE 11 *The foot of the perpendicular drawn from the origin to a plane is* $(4, -2, -5)$. *Find the equation of the plane in (i) vector form, (ii) Cartesian form.* **[CBSE 2007]**

SOLUTION Let $O(0, 0, 0)$ be the origin and $P(4, -2, -5)$ be the foot of the perpendicular drawn from O to the given plane.

(i) Clearly, the required plane passes through the point P and it is perpendicular to OP.

$\therefore \quad \vec{a} = (4\hat{i} - 2\hat{j} - 5\hat{k})$ and $\vec{n} = \overrightarrow{OP}\,(4\hat{i} - 2\hat{j} - 5\hat{k})$.

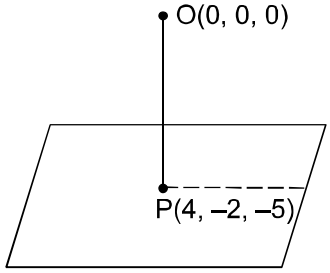
So, the required equation of the plane is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} = (4\hat{i} - 2\hat{j} - 5\hat{k})$$

$$= (4\hat{i} - 2\hat{j} - 5\hat{k}) \cdot (4\hat{i} - 2\hat{j} - 5\hat{k})$$

$$= (16 + 4 + 25) = 45.$$



Hence, the vector equation of the plane is

$$\vec{r} \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45 \qquad \text{... (i)}$$

(ii) Putting $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ in (i), we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45$$

$\Rightarrow$ $4x - 2y - 5z = 45$, which is the required equation of the plane in Cartesian form.

EXAMPLE 12 *Find the coordinates of the foot of the perpendicular drawn from the origin to the plane* $3y + 4z - 6 = 0$.

SOLUTION The equation of the given plane is

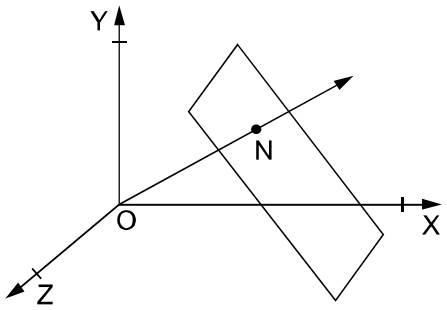
$$0x + 3y + 4z - 6 = 0. \qquad \text{... (i)}$$

D.r.'s of normal to the given plane are 0, 3, 4.

The equation of the line through the origin O and perpendicular to the plane (i) is given by

$$\frac{x}{0} = \frac{y}{3} = \frac{z}{4} = \lambda \text{ (say)}. \qquad \text{... (ii)}$$

The general point on (ii) is given by $N(0, 3\lambda, 4\lambda)$.



If this point lies on plane (i), we have

$$(0 \times 0) + (3 \times 3\lambda) + (4 \times 4\lambda) = 6 \;\Rightarrow\; 25\lambda = 6 \;\Rightarrow\; \lambda = \frac{6}{25}.$$

Hence, the foot of the perpendicular drawn from the origin to the given plane is $N\left(0, \frac{18}{20}, \frac{24}{25}\right)$.

EXAMPLE 13 *Find the coordinates of the point where the line* $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ *meets the plane* $x + y + 4z = 6$. **[CBSE 2006, '08]**

SOLUTION The equation of the given line is

$$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4} = \lambda \text{ (say)}. \qquad \text{... (i)}$$

The equation of the given plane is

$$x + y + 4z = 6 \qquad \text{... (ii)}$$

The general point on the line (i) is $(2\lambda - 1,\ 3\lambda - 2,\ 4\lambda - 3)$.

Let the given line (i) meet the given plane (ii) at the point $P(2\lambda - 1,\ 3\lambda - 2,\ 4\lambda - 3)$ for some value of λ.

Then, $(2\lambda - 1) + (3\lambda - 2) + 4(4\lambda - 3) = 6 \Rightarrow 21\lambda = 21 \Rightarrow \lambda = 1$.

Hence, the required point of intersection of the given line (i) and the given plane (ii) is $P(2 \times 1 - 1,\ 3 \times 1 - 2,\ 4 \times 1 - 3)$, i.e. $P(1, 1, 1)$.

EXAMPLE 14 *Find the coordinates of the point where the line through the points $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the XY-plane.* **[CBSE 2012]**

SOLUTION The equation of the line through the points $A(3, 4, 1)$ and $B(5, 1, 6)$ is given by

$$\frac{x-3}{(5-3)} = \frac{y-4}{(1-4)} = \frac{z-1}{(6-1)} \Rightarrow \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}. \qquad \text{... (i)}$$

The equation of the XY-plane is $z = 0$.

Since the line (i) meets the XY-plane, we have

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{0-1}{5} \Rightarrow \frac{x-3}{2} = \frac{-1}{5} \text{ and } \frac{y-4}{-3} = \frac{-1}{5}$$

$$\Rightarrow \quad x = \left(3 - \frac{2}{5}\right) = \frac{13}{5},\ y = \left(4 + \frac{3}{5}\right) = \frac{23}{5} \text{ and } z = 0.$$

Hence, the given line crosses the XY-plane at the point

$$P\left(\frac{13}{5}, \frac{23}{5}, 0\right).$$

EXAMPLE 15 *Find the distance of the point $P(-1, -5, -10)$ from the point of intersection of the line joining the points $A(2, -1, 2)$ and $B(5, 3, 4)$ with the plane $x - y + z = 5$.* **[CBSE 2014]**

SOLUTION The equation of the given plane is

$$x - y + z = 5 \qquad \text{... (i)}$$

The equation of the line AB is

$$\frac{x-2}{(5-2)} = \frac{y+1}{(3+1)} = \frac{z-2}{(4-2)} \Rightarrow \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda \text{ (say)}. \text{ ... (ii)}$$

For some value of λ, let the point $Q(3\lambda + 2,\ 4\lambda - 1,\ 2\lambda + 2)$ be the point of intersection of the plane (i) and the line (ii).

Then, $(3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5 \Rightarrow \lambda = 0$.

So, the required point Q is $Q(2, -1, 2)$.

$$\therefore \quad PQ = \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{3^2 + 4^2 + (12)^2} = \sqrt{9 + 16 + 144}$$

$$= \sqrt{169} = 13 \text{ units}.$$

Hence, the required distance is 13 units.

EXAMPLE 16 *Find the coordinates of the point where the line through* $(3, -4, -5)$ *and* $(2, -3, 1)$ *crosses the plane passing through the points* $(2, 2, 1), (3, 0, 1)$ *and* $(4, -1, 0)$. **[CBSE 2013]**

SOLUTION We know that the equation of a plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

$\therefore$ the equation of the plane passing through the points $A(2, 2, 1)$, $B(3, 0, 1)$ and $C(4, -1, 0)$ is given by

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4) = 0$$

$$\Rightarrow 2(x-2)+1\cdot(y-2)+1\cdot(z-1) = 0 \Rightarrow 2x+y+z = 7. \quad \text{... (i)}$$

The equation of the line passing through the points $P(3, -4, -5)$ and $Q(2, -3, 1)$ is given by

$$\frac{x-3}{(2-3)} = \frac{y+4}{(-3+4)} = \frac{z+5}{(1+5)} \Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}.$$

Any general point on line PQ is given as $(-\lambda+3, \lambda-4, 6\lambda-5)$.

For some value of λ, let the point $R(-\lambda+3, \lambda-4, 6\lambda-5)$ lie on the plane (i). Then,

$$2(-\lambda+3)+(\lambda-4)+(6\lambda-5) = 7 \Rightarrow 5\lambda = 10 \Rightarrow \lambda = 2.$$

Putting $\lambda = 2$, we get the point of intersection of the given line and the given plane as $R(-2+3, 2-4, 12-5)$, i.e., $R(1, -2, 7)$.

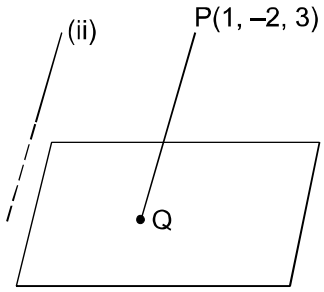
EXAMPLE 17 *Find the distance of the point* $(1, -2, 3)$ *from the plane* $x-y+z = 5$ *measured parallel to the line* $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$. **[CBSE 2013C]**

SOLUTION The equation of the given plane is

$$x-y+z = 5 \quad \text{.... (i)}$$

The equation of the given line is

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}. \quad \text{... (ii)}$$



The equation of the line passing through the point $P(1, -2, 3)$ and parallel to line (ii) is given by

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = \lambda \text{ (say)}. \quad \text{... (iii)}$$

The general point on line (iii) is $(2\lambda+1, 3\lambda-2, 6\lambda+3)$.

For some value of λ, let the point $Q(2\lambda+1,\ 3\lambda-2,\ -6\lambda+3)$ lie on plane (i). Then, we have

$$(2\lambda+1)-(3\lambda-2)+(-6\lambda+3)=5 \ \Rightarrow\ -7\lambda=-1 \ \Rightarrow\ \lambda=\frac{1}{7}.$$

So, the coordinates of Q are

$$Q\left(\frac{2}{7}+1,\ \frac{3}{7}-2,\ \frac{-6}{7}+3\right), \text{ i.e., } Q\left(\frac{9}{7},\ \frac{-11}{7},\ \frac{15}{7}\right).$$

$$\text{Distance } PQ=\sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{-6}{7}\right)^2}=\frac{1}{7}\sqrt{4+9+36}$$

$$=\frac{1}{7}\times\sqrt{49}=\left(\frac{1}{7}\times 7\right)=1 \text{ unit.}$$

Hence, the required distance is 1 unit.

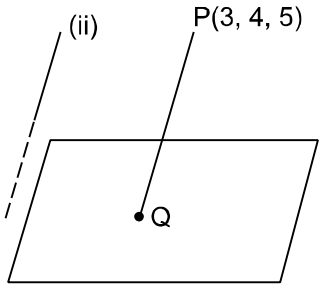
EXAMPLE 18 *Find the distance of the point* $(3, 4, 5)$ *from the plane* $x+y+z=2$, *measured parallel to the line* $2x=y=z$. **[CBSE 2012C]**

SOLUTION The equation of the given plane is

$$x+y+z=2. \qquad \text{... (i)}$$

The equation of the given line is

$$2x=y=z \ \Rightarrow\ \frac{x}{1}=\frac{y}{2}=\frac{z}{2}. \qquad \text{... (ii)}$$



The equation of the line passing through the point $P(3, 4, 5)$ and parallel to the given line (ii) is

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=\lambda \text{ (say).} \qquad \text{... (iii)}$$

The general point on line (iii) is $(\lambda+3,\ 2\lambda+4,\ 2\lambda+5)$.

For some value of λ, let the point $Q(\lambda+3,\ 2\lambda+4,\ 2\lambda+5)$ lie on the plane (i). Then,

$$(\lambda+3)+(2\lambda+4)+(2\lambda+5)=2$$

$$\Rightarrow\ 5\lambda=-10 \ \Rightarrow\ \lambda=-2.$$

$\therefore$ coordinates of Q are $Q(-2+3,\ -4+4,\ -4+5)$, i.e., $Q(1,\ 0,\ 1)$.

$$\text{Distance } PQ=\sqrt{(3-1)^2+(4-0)^2+(5-1)^2}$$

$$=\sqrt{2^2+4^2+4^2}=\sqrt{4+16+16}$$

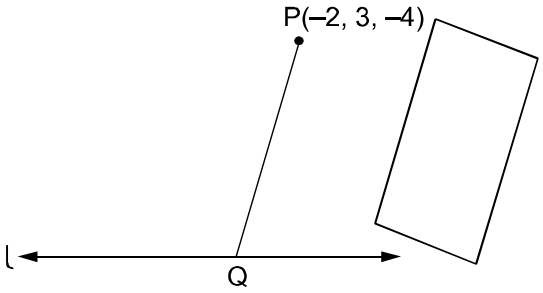
$$=\sqrt{36}=6 \text{ units.}$$

Hence, the required distance is 6 units.

EXAMPLE 19 *Find the distance of the point* $(-2, 3, -4)$ *from the line* $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$, *measured parallel to the plane* $4x + 12y - 3z + 1 = 0$. **[CBSE 2008]**

SOLUTION The equation of the given plane is

$$4x + 12y - 3z + 1 = 0. \quad \text{... (i)}$$



Let $P(-2, 3, -4)$ be the given point.

Let l be the given line whose equation is

$$\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5} = \lambda \text{ (say)}. \quad \text{... (ii)}$$

The general point on this line is $\left(3\lambda - 2, \frac{4\lambda - 3}{2}, \frac{5\lambda - 4}{3}\right)$.

For some value of λ, let the point $Q\left(3\lambda - 2, \frac{4\lambda - 3}{2}, \frac{5\lambda - 4}{3}\right)$ lie on the line (ii) such that PQ is parallel to the given plane (i).

D.r.'s of PQ are $(3\lambda - 2 + 2)$, $\left(\frac{4\lambda - 3}{2} - 3\right)$, $\left(\frac{5\lambda - 4}{3} + 4\right)$,

i.e., $3\lambda, \frac{4\lambda - 9}{2}, \frac{5\lambda + 8}{3}$.

D.r.'s of the normal to the given plane are 4, 12, –3.

Now, PQ is parallel to the given plane (i).

$\Rightarrow$ PQ is perpendicular to the normal to the the plane (i).

$$\Rightarrow (4 \times 3\lambda) + 12 \times \frac{(4\lambda - 9)}{2} - 3 \times \frac{(5\lambda - 8)}{3} = 0$$

$$\Rightarrow 12\lambda + (24\lambda - 54) - (5\lambda - 8) = 0 \Rightarrow 31\lambda = 62 \Rightarrow \lambda = 2.$$

$\therefore$ coordinates of Q are $Q\left(4, \frac{5}{2}, 2\right)$

$$\Rightarrow PQ = \sqrt{(4+2)^2 + \left(\frac{5}{2} - 3\right)^2 + (2+4)^2} = \sqrt{6^2 + \left(\frac{-1}{2}\right)^2 + 6^2}$$

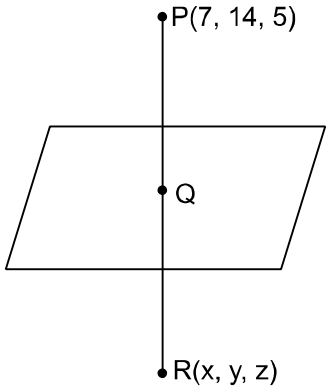
$$= \sqrt{72 + \frac{1}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2} \text{ units} = 8.5 \text{ units}.$$

Hence, the required distance of the given point from the given line is 8.5 units.

EXAMPLE 20 *Find the length and the foot of the perpendicular from the point* $P(7, 14, 5)$ *to the plane* $2x + 4y - z = 2$. *Also, find the image of the point* P *in the plane.* **[CBSE 2012]**

SOLUTION The equation of the given plane is

$$2x + 4y - z = 2 \qquad \ldots (i)$$



The d.r.'s of the normal to this plane are 2, 4, –1.

The equation of the line passing through the point $P(7, 14, 5)$ and perpendicular to the given plane (i) is given by

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = \lambda \text{ (say)}$$

A general point on this line is $(2\lambda + 7, 4\lambda + 14, -\lambda + 5)$.

For some value of λ, let the point $Q(2\lambda + 7, 4\lambda + 14, -\lambda + 5)$ lie on the plane (i). Then,

$$2(2\lambda + 7) + 4(4\lambda + 14) - (-\lambda + 5) = 2 \Rightarrow 21\lambda = -63 \Rightarrow \lambda = -3.$$

$\therefore$ the coordinates of the foot of the perpendicular PQ are

$$Q[2\times(-3) + 7, 4\times(-3) + 14, -(-3) + 5], \text{ i.e., } Q(1, 2, 8).$$

$$\therefore \quad PQ = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2}$$

$$= \sqrt{6^2 + (12)^2 + (-3)^2} = \sqrt{36 + 144 + 9}$$

$$= \sqrt{189} = 3\sqrt{21} \text{ units.}$$

Let $R(x, y, z)$ be the image of P in the plane (i).

Then, Q is the midpoint of PR.

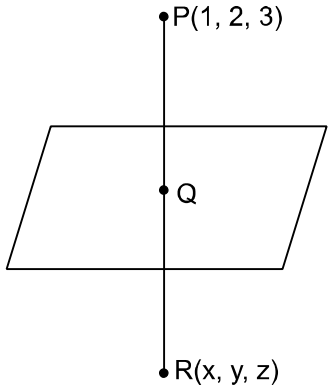
$$\therefore \quad \frac{7+x}{2} = 1, \frac{14+y}{2} = 2 \text{ and } \frac{5+z}{2} = 8 \Rightarrow x = -5, y = -10, z = 11.$$

Hence, the image of $P(7, 14, 5)$ in the given plane is $R(-5, -10, 11)$.

EXAMPLE 21 *Find the image of the point* $(1, 2, 3)$ *in the plane* $x + 2y + 4z = 38$.

SOLUTION The equation of the given plane is

$$x + 2y + 4z = 38. \qquad \ldots (i)$$



The d.r.'s of the normal to this plane are 1, 2, 4.

The equation of the line passing through the point $P(1, 2, 3)$ and perpendicular to the given plane (i) is given by

$$\frac{(x-1)}{1} = \frac{(y-2)}{2} = \frac{(z-3)}{4} = \lambda \text{ (say).}$$

A general point on this line is $(\lambda + 1, 2\lambda + 2, 4\lambda + 3)$.

For some value of λ, let the point $Q(\lambda + 1, 2\lambda + 2, 4\lambda + 3)$ lie on the plane (i). Then,

$(\lambda+1)+2(2\lambda+2)+4(4\lambda+3)=38 \Rightarrow 21\lambda=21 \Rightarrow \lambda=1.$

$\therefore$ the coordinates of the foot of the perpendicular PQ are $Q(1+1, 2+2, 4+3)$, i.e., $Q(2, 4, 7)$.

Let $R(x, y, z)$ be the image of $P(1, 2, 3)$ in the given plane.

Then, Q is the midpoint of PR.

$\therefore \quad \frac{1+x}{2}=2, \frac{2+y}{2}=4$ and $\frac{3+z}{2}=7 \Rightarrow x=3, y=6, z=11.$

Hence, the image of $P(1, 2, 3)$ in the given plane is $R(3, 6, 11)$.

EXERCISE 28B

1. Find the vector and Cartesian equations of a plane which is at a distance of 5 units from the origin and which has $\hat{k}$ as the unit vector normal to it.
2. Find the vector and Cartesian equations of a plane which is at a distance of 7 units from the origin and whose normal vector from the origin is $(3\hat{i}+5\hat{j}-6\hat{k})$.
3. Find the vector and Cartesian equations of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and whose normal vector from the origin is $(2\hat{i}-3\hat{j}+4\hat{k})$.
4. Find the vector and Cartesian equations of a plane which is at a distance of 6 units from the origin and which has a normal with direction ratios 2, –1, –2.
5. Find the vector and Cartesian equations of a plane which passes through the point (1, 4, 6) and normal vector to the plane is $(\hat{i}-2\hat{j}+\hat{k})$.
6. Find the length of perpendicular from the origin to the plane $\vec{r}\cdot(3\hat{i}-12\hat{j}-4\hat{k})+39=0$. Also write the unit normal vector from the origin to the plane.
7. Find the Cartesian equation of the plane whose vector equation is $\vec{r}\cdot(3\hat{i}+5\hat{j}-9\hat{k})=8$.
8. Find the vector equation of a plane whose Cartesian equation is $5x-7y+2z+4=0$.
9. Find a unit vector normal to the plane $x-2y+2z=6$.
10. Find the direction cosines of the normal to the plane $3x-6y+2z=7$.
11. For each of the following planes, find the direction cosines of the normal to the plane and the distance of the plane from the origin:
    (i) $2x+3y-z=5$ (ii) $z=3$ (iii) $3y+5=0$

12. Find the vector and Cartesian equations of the plane passing through the point (2, –1, 1) and perpendicular to the line having direction ratios 4, 2, –3.

13. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane (i) $2x + 3y + 4z - 12 = 0$ (ii) $5y + 8 = 0$.

14. Find the length and the foot of perpendicular drawn from the point (2, 3, 7) to the plane $3x - y - z = 7$.

15. Find the length and the foot of the perpendicular drawn from the point (1, 1, 2) to the plane $\vec{r} \cdot (2\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$.

16. From the point $P(1, 2, 4)$, a perpendicular is drawn on the plane $2x + y - 2z + 3 = 0$. Find the equation, the length and the coordinates of the foot of the perpendicular. **[CBSE 2008]**

17. Find the coordinates of the foot of the perpendicular and the perpendicular distance from the point $P(3, 2, 1)$ to the plane $2x - y + z + 1 = 0$.
Find also the image of the point P in the plane. **[CBSE 2010]**

18. Find the coordinates of the image of the point $P(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$. **[CBSE 2008]**

19. Find the point where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z = 1$.

20. Find the coordinates of the point where the line through (3, –4, –5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.

21. Find the distance of the point (2, 3, 4) from the plane $3x + 2y + 2z + 5 = 0$, measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$. **[CBSE 2007, '09C]**

22. Find the distance of the point (0, –3, 2) from the plane $x + 2y - z = 1$, measured parallel to the line $\frac{x+1}{3} = \frac{y+1}{2} = \frac{z}{3}$.

23. Find the equation of the line passing through the point $P(4, 6, 2)$ and the point of intersection of the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7}$ and the plane $x + y - z = 8$.
[CBSE 2008]

24. Show that the distance of the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is 13 units.

25. Find the distance of the point $A(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. **[CBSE 2011, '14]**

HINT: Convert the equations of the line and the plane to Cartesian form.

26. Prove that the normals to the planes $4x+11y+2z+3=0$ and $3x-2y+5z=8$ are perpendicular to each other.

27. Show that the line $\vec{r}=(2\hat{i}-2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+4\hat{k})$ is parallel to the plane $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=7$.

28. Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from the origin and the normal to which is equally inclined to the coordinate axes.

29. A vector $\vec{n}$ of magnitude 8 units is inclined to the x-axis at 45°, y-axis at 60° and an acute angle with the z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to $\vec{n}$, find its equation in vector form.

30. Find the vector equation of a line passing through the point $(2\hat{i}-3\hat{j}-5\hat{k})$ and perpendicular to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+5\hat{k})+2=0$.
Also, find the point of intersection of this line and the plane. **[CBSE 2006C]**

ANSWERS (EXERCISE 28B)

1. $\vec{r}\cdot\hat{k}=5, z=5$ **2.** $\vec{r}\cdot(3\hat{i}+5\hat{j}-6\hat{k})=7\sqrt{70},\ 3x+5y-6z=7\sqrt{70}$

3. $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=6,\ 2x-3y+4z=6$

4. $\vec{r}\cdot(2\hat{i}-\hat{j}-2\hat{k})=18,\ 2x-y-2z=18$

5. $\vec{r}\cdot(-\hat{i}+2\hat{j}-\hat{k})=1,\ x-2y+z+1=0$

6. $p=3, \hat{n}=\left(\frac{-3}{13}\hat{i}+\frac{12}{13}\hat{j}+\frac{4}{13}\hat{k}\right)$ **7.** $3x+5y+9z=8$

8. $\vec{r}\cdot(-5\hat{i}+7\hat{j}-2\hat{k})=4$ **9.** $\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$ **10.** $\frac{3}{7}, \frac{-6}{7}, \frac{2}{7}$

11. (i) $\left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right), \frac{5}{\sqrt{14}}$ (ii) $(0, 0, 1), 3$ (iii) $(0, -1, 0), \frac{5}{3}$

12. $\vec{r}\cdot(4\hat{i}+2\hat{j}-3\hat{k})=3,\ 4x+2y-3z=3$

13. (i) $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$ (ii) $\left(0, \frac{-8}{5}, 0\right)$ **14.** $\sqrt{11}$ units, $(5, 2, 6)$

15. $\frac{13\sqrt{6}}{12}$ units, $\left(\frac{-1}{12}, \frac{25}{12}, \frac{-1}{6}\right)$ **16.** $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-4}{-2}; \frac{1}{3}$ unit; $\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

17. $(1, 3, 0), \sqrt{6}$ units, $(-1, 4, -1)$ **18.** $(-3, 5, 2)$ **19.** $(3, -1, 1)$ **20.** $(1, -2, 7)$

21. 7 units **22.** 10 units **23.** $\frac{x-4}{1}=\frac{y-6}{2}=\frac{z-2}{2}$ **25.** 13 units

28. $x+y+z=9$ **29.** $\vec{r}\cdot(\sqrt{2}\hat{i}+\hat{j}+\hat{k})=2$

30. $\vec{r}\cdot(2\hat{i}-3\hat{j}-5\hat{k})+\lambda(6\hat{i}-3\hat{j}+5\hat{k}),\left(\frac{76}{35},\frac{-108}{35},\frac{-34}{4}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28B)

6. Given equation in standard form is $\vec{r}\cdot(-3\hat{i}+12\hat{j}+4\hat{k})=39.$

This is $\vec{r}\cdot\vec{n}=p,$ where $\vec{n}=(-3\hat{i}+12\hat{j}+4\hat{k})$ and $p=39.$

$$|\vec{n}|=\sqrt{(-3)^2+(12)^2+4^2}=\sqrt{9+144+16}=\sqrt{169}=13.$$

$$\therefore\quad \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{p}{|\vec{n}|}=\frac{39}{13}=3.$$

$$\vec{r}\cdot\left(\frac{-3}{13}\hat{i}+\frac{12}{13}\hat{j}+\frac{4}{13}\hat{k}\right)=3.$$

Hence, $p=3$ and $\hat{n}=\frac{-3}{13}\hat{i}+\frac{12}{13}\hat{j}+\frac{4}{13}\hat{k}.$

9. Given equation in vector form is $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=6.$

Here, $|\hat{i}-2\hat{j}+2\hat{k}|=\sqrt{1^2+(-2)^2+2^2}=\sqrt{9}=3.$

$$\vec{r}\cdot\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)=\frac{6}{3}=2.$$

Required unit vector $=\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right).$

10. Given equation in vector form is $\vec{r}\cdot(3\hat{i}-6\hat{j}+2\hat{k})=7.$

$$|3\hat{i}-6\hat{j}+2\hat{k}|=\sqrt{3^2+(-6)^2+2^2}=\sqrt{49}=7.$$

$$\therefore\quad \vec{r}\cdot\left(\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}\right)=1.$$

$\therefore$ d.c.'s of the normal to the plane are $\frac{3}{7},\frac{-6}{7},\frac{2}{7}.$

11. (i) Given equation in vector form is $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})=5.$

$$|2\hat{i}+3\hat{j}-\hat{k}|=\sqrt{2^2+3^2+(-1)^2}=\sqrt{14}.$$

$$\therefore\quad \vec{r}\cdot\left(\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}-\frac{1}{\sqrt{14}}\hat{k}\right)=\frac{5}{\sqrt{14}}.$$

d.c.'s of $\hat{n}$ are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$ and $p = \frac{5}{\sqrt{14}}$.

(ii) Given equation in vector form is $\vec{r} \cdot (0\hat{i} + 0\hat{j} + 1\hat{k}) = 3$.

This is $\vec{r} \cdot \hat{n} = p$.

D.c.'s of $\hat{n}$ are 0, 0, 1 and $p = 3$.

(iii) Given equation in vector form is $\vec{r} \cdot (0\hat{i} - 1 \cdot \hat{j} + 0 \cdot \hat{k}) = \frac{5}{3}$ $\left[\because -y = \frac{5}{3}\right]$.

Clearly, $|0\hat{i} - 1 \cdot \hat{j} + 0 \cdot \hat{k}| = \sqrt{0^2 + (-1)^2 + 0^2} = 1$.

$\therefore$ $\vec{r} \cdot \hat{n} = p$, where $\hat{n} = (0\hat{i} - 1 \cdot \hat{j} + 0\hat{k})$ and $p = \frac{5}{3}$.

D.c.'s of $\hat{n}$ are 0, –1, 0 and $p = \frac{5}{3}$.

12. Let the equation of the plane be $a(x-2) + b(y+1) + c(z-1) = 0$, where $a = 4$, $b = 2$ and $c = -3$.

So, the required equation is

$$4(x-2) + 2(y+1) - 3(z-1) = 0 \Rightarrow 4x + 2y - 3z = 3.$$

In vector form, we have $\vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3$.

13. (i) Given equation is $2x + 3y + 4z = 12$. ... (i)

D.r.'s of normal to the plane are 2, 3, 4.

Equation of the line through $O(0, 0, 0)$ and perpendicular to plane (i) is

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4} = \lambda \text{ (say)}. \quad \text{... (ii)}$$

A general point on (ii) is $N(2\lambda, 3\lambda, 4\lambda)$.

If this point lies on plane (i), then

$$(2 \times 2\lambda) + (3 \times 3\lambda) + (4 \times 4\lambda) = 12 \Rightarrow 29N = 12 \Rightarrow \lambda = \frac{12}{29}.$$

Hence, the required point is $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$.

(ii) The given plane is $0x - 5y + 0z = 8$. ... (i)

D.r.'s of normal to the plane are 0, –5, 0.

Equation of the line through $O(0, 0, 0)$ and perpendicular to plane (i) are

$$\frac{x}{0} = \frac{y}{-5} = \frac{z}{0} = \lambda \text{ (say)}. \quad \text{... (ii)}$$

A general point on this line is $N(0, -5\lambda, 0)$.

If this point lies on plane (i), we have

$$-5 \times (-5\lambda) = 8 \Rightarrow 25\lambda = 8 \Rightarrow \lambda = \frac{8}{25}.$$

Hence, the required point is $N\left(0, \frac{-8}{5}, 0\right)$.

21. Equation of a line through $A(2, 3, 4)$ and parallel to the given line is

$$\frac{x-2}{3}=\frac{y-3}{6}=\frac{z-4}{2}=\lambda \text{ (say).} \qquad \text{... (i)}$$

A general point on line (i) is $N(3\lambda+2, 6\lambda+3, 2\lambda+4)$.

For some value of λ, it lies on the plane $3x+2y+2z+5=0$.

$\therefore$ $3(3\lambda+2)+2(6\lambda+3)+2(2\lambda+4)+5=0 \Rightarrow 25\lambda=-25 \Rightarrow \lambda=-1$.

$\therefore$ the coordinates of N are $(-3+2, -6+3, -2+4)$, i.e., $(-1, -3, 2)$.

Now, find $AN = 7$ units.

24. Given line is $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=\lambda$ (say)

A general point on this line is $P(3\lambda+2, 4\lambda-1, 12\lambda+2)$.

If this point lies on the plane $x-y+z=5$, then

$$(3\lambda+2)-(4\lambda-1)+(12\lambda+2)=5 \Rightarrow 11\lambda=0 \Rightarrow \lambda=0.$$

$\therefore$ point P is $P(2, -1, 2)$ and the other point is $Q(-1, -5, -10)$

Find $PQ = 13$ units.

25. The Cartesian equation of the plane is $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=\lambda$ (say).

A general point on this line is $P(3\lambda+2, 4\lambda-1, 2\lambda+2)$.

The Cartesian equation of the given plan is $x-y+z=5$.

If this point lies on the given plane, we have

$$(3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5 \Rightarrow \lambda=0.$$

$\therefore$ point P is $P(2, -1, 2)$ and the other point is $Q(-1, -5, -10)$.

$\therefore$ $PQ = 13$ units.

26. $\vec{n_1}=(4\hat{i}+11\hat{j}+2\hat{k})$ and $\vec{n_2}=(3\hat{i}-2\hat{j}+5\hat{k})$.

$\therefore$ $\vec{n_1}\cdot\vec{n_2}=(4\times 3)+11\times(-2)+2\times 5=0$.

Hence, $\vec{n_1}\perp\vec{n_2}$.

27. D.r.'s of the line are 1, –1, 4.

D.r.'s of normal to the plane are 1, 5, 1.

$\therefore$ $(1\times 1)+(-1)\times 5+(4\times 1)=0$.

So, the given line is perpendicular to the normal of the given plane.

Hence, the given line is parallel to the given plane.

28. Let the required equation of the plane be $\vec{r}\cdot\hat{n}=p$, where $p=3\sqrt{3}$.

Let $\hat{n}=(\cos\alpha)\hat{i}+(\cos\alpha)\hat{j}+(\cos\alpha)\hat{k}$, where α is acute.

Then, $\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1 \Rightarrow 3\cos^2\alpha=1 \Rightarrow \cos^2\alpha=\frac{1}{3}$

$$\Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}.$$

$\therefore$ the required equation is

$$\vec{r}\cdot\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)=3\sqrt{3}.$$

Hence, $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=9 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})=9$

$\Rightarrow x+y+z=9.$

29. We know that $\hat{n}=\dfrac{\vec{n}}{|\vec{n}|}=(l\hat{i}+m\hat{j}+n\hat{k}).$

Here $l=\cos 45°=\dfrac{1}{\sqrt{2}}$, $m=\cos 60°=\dfrac{1}{2}$ and let $n=\cos\gamma$. Then,

$$l^2+m^2+n^2=1 \Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\gamma=1$$

$$\Rightarrow \cos^2\gamma=\frac{1}{4}$$

$$\Rightarrow \cos\gamma=\frac{1}{2}.$$

$\therefore \quad \vec{n}=|\vec{n}|(l\hat{i}+m\hat{j}+n\hat{k}) \Rightarrow \vec{n}=8\left(\dfrac{1}{\sqrt{2}}\hat{i}+\dfrac{1}{2}\hat{j}+\dfrac{1}{2}\hat{k}\right)=(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k}).$

So, the equation of the plane is

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n} \Rightarrow \vec{r}\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k})=(\sqrt{2}\hat{i}-\hat{j}+\hat{k})\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k})$$

$$\Rightarrow \vec{r}\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k})=(8-4+4)=8$$

$$\Rightarrow \vec{r}\cdot(\sqrt{2}\hat{i}+\hat{j}+\hat{k})=2.$$

30. Clearly, the required line passes through the point $(2\hat{i}-3\hat{j}-5\hat{k})$ and is parallel to the normal of the given plane, which is $(6\hat{i}-3\hat{j}+5\hat{k})$.

So, the required vector equation is

$$\vec{r}=(2\hat{i}-3\hat{j}-5\hat{k})+\lambda(6\hat{i}-3\hat{j}+5\hat{k}).$$

The Cartesian equation of the line is $\dfrac{x-2}{6}=\dfrac{y+3}{-3}=\dfrac{z+5}{5}=k.$

A general point on this line is $P(6k+2,-3k-3,5k-5)$.

For some particular value of k, let the line cut the plane $6x-3y+5z+2=0$.

Then, $6(6k+2)-3(-3k-3)+5(5k-5)+2=0$

$\Rightarrow \quad (36k+9k+25k)=2 \Rightarrow 70k=2$

$\Rightarrow \quad k=\dfrac{1}{35}.$

$\therefore$ required point of intersection of the line and plane is

$$P\left(\frac{6}{35}+2,\frac{-3}{35}-3,\frac{1}{7}-5\right), \text{ i.e., } P\left(\frac{76}{35},\frac{-108}{35},\frac{-34}{7}\right).$$

DISTANCE OF A POINT FROM A PLANE

1. VECTOR FORM

(i) Let p be the length of perpendicular drawn from a point P with position vector $\vec{a}$ to the plane $\vec{r} \cdot \vec{n} = q$. Then,

$$p = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}.$$

(ii) Let p be the length of perpendicular drawn from the origin to the plane $\vec{r} \cdot \vec{n} = q$. Then,

$$p = \frac{|q|}{|\vec{n}|}.$$

2. CARTESIAN FORM

(i) Let p be the length of perpendicular drawn from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$. Then,

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

(ii) Let p be the length of perpendicular drawn from the origin to the plane $ax + by + cz + d = 0$. Then,

$$p = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$$

3. DISTANCE BETWEEN PARALLEL PLANES

Let $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ be two parallel planes.
Let $P(x_1, y_1, z_1)$ be any point on the first plane.
Then, the length of perpendicular from $P(x_1, y_1, z_1)$ to the other plane is the required distance between these planes.

SOLVED EXAMPLES

EXAMPLE 1 *Find the distance of the point* $(\hat{i} + 2\hat{j} - 3\hat{k})$ *from the plane* $\vec{r} \cdot (2\hat{i} - 5\hat{j} - \hat{k}) = 4$.

SOLUTION We know that the perpendicular distance of a point with position vector $\vec{a}$, from the plane $\vec{r} \cdot \vec{n} = q$ is given by $p = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}$.

Here $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$, $\vec{n} = (2\hat{i} - 5\hat{j} - \hat{k})$ and $q = 4$.

Hence, the required distance is given by

$$p = \frac{|(\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (2\hat{i} - 5\hat{j} - \hat{k}) - 4|}{\sqrt{2^2 + (-5)^2 + (-1)^2}}$$

$$= \frac{|2 - 10 + 3 - 4|}{\sqrt{30}} = \frac{|-9|}{\sqrt{30}} = \frac{9}{\sqrt{30}} \times \frac{\sqrt{30}}{\sqrt{30}}$$

$$= \frac{9}{30}\sqrt{30} = \frac{3}{10}\sqrt{30} \text{ units.}$$

Hence, the required distance is $\frac{3}{10}\sqrt{30}$ units.

EXAMPLE 2 *Find the distance of the point* $(2, 3, 4)$ *from the plane*

$$\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) + 11 = 0.$$

SOLUTION We know that the perpendicular distance of a point with position vector $\vec{a}$, from the plane $\vec{r} \cdot \vec{n} = q$ is given by

$$p = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}.$$

Here $\vec{a} = (2\hat{i} + 3\hat{j} + 4\hat{k})$, $\vec{n} = (3\hat{i} - 6\hat{j} + 2\hat{k})$ and $q = -11$.

$\therefore$ the required distance is given by

$$p = \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) - (-11)|}{\sqrt{3^2 + (-6)^2 + 2^2}}$$

$$= \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) + 11|}{\sqrt{9 + 36 + 4}}$$

$$= \frac{|(2 \times 3) + 3 \times (-6) + (4 \times 2) + 11|}{\sqrt{49}}$$

$$= \frac{|6 - 18 + 8 + 11|}{7} = \frac{7}{7} = 1 \text{ unit.}$$

Hence, the required distance is 1 unit.

EXAMPLE 3 *Find the distance of the point* $(2, 3, -5)$ *from the plane* $x + 2y - 2z = 9$.

SOLUTION Required distance

= Length of perpendicular from (2, 3, –5) to the plane

$$x + 2y - 2z - 9 = 0$$

$$= \frac{|2 + 2 \times 3 - 2 \times (-5) - 9|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{|2 + 6 + 10 - 9|}{\sqrt{9}}$$

$$= \frac{9}{3} = 3 \text{ units.}$$

Hence, the required distance is 3 units.

EXAMPLE 4 *If a plane has intercepts a, b and c on the coordinate axes and is at a distance of p untis from the origin, prove that* $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$.

SOLUTION We know that the equation of the plane which makes intercepts a, b and c on the axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0. \quad \text{... (i)}$$

It is given that p is the length of perpendicular drawn from $O(0, 0, 0)$ to the plane (i).

$$\therefore \quad p=\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{1}{p} \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}.$$

Hence $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$.

EXAMPLE 5 *Find the distance of the point P(6, 5, 9) from the plane determined by the points A(3, −1, 2), B(5, 2, 4) and C(−1, −1, 6).* **[CBSE 2009, '10, '12]**

SOLUTION Here

$$(x_1 = 3, y_1 = -1, z_1 = 2), (x_2 = 5, y_2 = 2, z_2 = 4)$$

and $(x_3 = -1, y_3 = -1, z_3 = 6)$.

The equation of the plane ABC is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix}=0$$

$$\Rightarrow (x-3)(12-0)-(y+1)(8+8)+(z-2)(0+12)=0$$

$$\Rightarrow 12(x-3)-16(y+1)+12(z-2)=0$$

$$\Rightarrow 12x-16y+12z-76=0 \Rightarrow 3x-4y+3z-19=0. \quad \text{... (i)}$$

Distance of the point $P(6, 5, 9)$ from the plane $3x-4y+3z-19=0$

= length of perpendicular from $P(6, 5, 9)$ to the plane

$$3x-4y+3z-19=0$$

$$=\frac{|(3\times 6)-(4\times 5)+(3\times 9)-19|}{\sqrt{3^2+(-4)^2+3^2}}=\frac{|18-20+27-19|}{\sqrt{34}}$$

$$= \frac{6}{\sqrt{34}} \times \frac{\sqrt{34}}{\sqrt{34}} = \frac{6}{34}\sqrt{34} = \frac{3}{17}\sqrt{34} \text{ units.}$$

Hence, the required distance is $\frac{3}{17}\sqrt{34}$ units.

EXAMPLE 6 *Find the distance between the parallel planes* $2x - y + 3z + 4 = 0$ *and* $6x - 3y + 9z - 3 = 0$.

SOLUTION Let $P(x_1\ y_1, z_1)$ be any point on the plane $2x - y + 3z + 4 = 0$.

Then, $2x_1 - y_1 + 3z_1 = -4$. ... (i)

Length of perpendicular from $P(x_1, y_1, z_1)$ to the plane $6x - 3y + 9z = 0$ is given by

$$p = \frac{|6x_1 - 3y_1 + 9z_1 - 3|}{\sqrt{6^2 + (-3)^2 + 9^2}} = \frac{|3(2x_1 - y_1 + 3z_1) - 3|}{\sqrt{126}}$$

$$= \frac{|3 \times (-4) - 3|}{3\sqrt{14}} \text{ [using (i)]}$$

$$= \frac{|-15|}{3\sqrt{14}} = \frac{15}{3\sqrt{14}} \times \frac{\sqrt{14}}{\sqrt{14}} = \frac{5}{14}\sqrt{14} \text{ units.}$$

Hence, the distance between the given planes is $\frac{5}{14}\sqrt{14}$ units.

EXAMPLE 7 *Find the distance between the parallel planes* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ *and* $\vec{r} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$.

SOLUTION Taking $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, the Cartesian equations of the given planes are

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$$

$$\Rightarrow \quad 2x - 3y + 6z - 5 = 0 \quad \text{... (i)}$$

and $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$

$$\Rightarrow \quad 6x - 9y + 18z + 20 = 0 \quad \text{... (ii)}$$

Let $P(x_1, y_1, z_1)$ be any point on (i). Then, $2x_1 - 3y_1 + 6z_1 = 5$ ... (iii)

Distance between the given planes

= length of the perpendicular from $P(x_1, y_1, z_1)$ to the plane (ii)

$$= \frac{|6x_1 - 9y_1 + 18z_1 + 20|}{\sqrt{6^2 + (-9)^2 + (18)^2}} = \frac{|3(2x_1 - 3y_1 + 6z_1) + 20|}{\sqrt{36 + 81 + 324}}$$

$$= \frac{|(3 \times 5) + 20|}{\sqrt{441}} \text{ [using (iii)]}$$

$$= \frac{35}{21} = \frac{5}{3} \text{ units.}$$

Hence, the distance between the given planes is $\frac{5}{3}$ units.

EXAMPLE 8 *Find the equations of planes parallel to the plane* $x - 2y + 2z = 3$ *which are at a unit distance from the point* $(1, 2, 3)$.

SOLUTION The equation of the given plane is $x - 2y + 2z - 3 = 0$.

Let the required plane parallel to the given plane be

$x - 2y + 2z + \lambda = 0$ for some real number λ ... (i)

Now, the plane (i) is at a unit distance from the point (1, 2, 3).

$$\therefore \quad \frac{|1-(2\times 2)+(2\times 3)+\lambda|}{1^2+(-2)^2+2^2} = 1 \Rightarrow |\lambda + 3| = 3$$

$$\Rightarrow \lambda + 3 = 3 \text{ or } -(\lambda + 3) = 3$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = -6.$$

Hence, the equations of the required planes are

$x - 2y + 2z = 0$ and $x - 2y + 2z - 6 = 0$.

EXAMPLE 9 *Find the equation of the plane passing through the point* $(2, -3, 5)$ *and parallel to the plane* $3x - 7y - 2z = 5$. *Also, find the distance between the two planes.*

SOLUTION The given plane is $3x - 7y - 2z - 5 = 0$. ... (i)

Let the required plane parallel to the given plane be

$3x - 7y - 2z + \lambda = 0$ for some real number λ. ... (ii)

Since the plane (ii) passes through the point (2, –3, 5), we have

$$(3\times 2) - 7\times(-3) - (2\times 5) + \lambda = 0 \Rightarrow (6 + 21 - 10) + \lambda = 0$$

$$\Rightarrow \lambda = -17.$$

$\therefore$ the equation of the required plane is $3x - 7y - 2z - 17 = 0$. ... (iii)

Now, the planes (i) and (ii) are parallel and the point (2, –3, 5) lies on (iii).

$\therefore$ distance between the two planes

= distance between the point (2, –3, 5) and the plane (i)

$$= \frac{|(3\times 2) - 7\times(-3) - (2\times 5) - 5|}{\sqrt{3^2 + (-7)^2 + (-2)^2}}$$

$$= \frac{|6 + 21 - 10 - 5|}{\sqrt{9 + 49 + 4}}$$

$$= \frac{12}{\sqrt{62}} \times \frac{\sqrt{62}}{\sqrt{62}} = \frac{12}{62}\sqrt{62}$$

$$= \frac{6}{31}\sqrt{62} \text{ units.}$$

Hence, the distance between the two planes is $\frac{6}{31}\sqrt{62}$ units.

EXAMPLE 10 *Find the Cartesian as well as the vector equations of the planes through the intersection of the planes* $\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$ *and* $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$ *which are at a unit distance from the origin.* **[CBSE 2005]**

SOLUTION The equations of the given planes are

$$\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+6\hat{j})+12=0$$

$$\Rightarrow 2x+6y+12=0 \Rightarrow x+3y+6=0 \quad \text{... (i)}$$

and, $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$$

$$\Rightarrow 3x-y+4z=0. \quad \text{... (ii)}$$

Equation of any plane through the intersection of these planes is

$$(x+3y+4)+\lambda(3x-y+4z)=0 \text{ for some real number } \lambda$$

$$\Rightarrow (1+3\lambda)x+(3-\lambda)y+4\lambda z+6=0 \quad \text{... (iii)}$$

Now, plane (iii) is at a unit distance from $O(0, 0, 0)$.

$$\therefore \quad \frac{|0+0+0+6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(4\lambda)^2}}=1$$

$$\Rightarrow (1+3\lambda)^2+(3-\lambda)^2+(4\lambda)^2=36$$

$$\Rightarrow (1+9\lambda^2+6\lambda)+(9+\lambda^2-6\lambda)+16\lambda^2=36$$

$$\Rightarrow 26\lambda^2=26 \Rightarrow \lambda^2=1 \Rightarrow \lambda=1 \text{ or } \lambda=-1.$$

Putting $\lambda=1$ in (iii), we get

$$4x+2y+4z+6=0 \Rightarrow 2x+y+2z+3=0.$$

Putting $\lambda=-1$ in (iii) we get

$$-2x+4y-4z+6=0 \Rightarrow x-2y+2z-3=0.$$

Hence, the required equations of the planes in Cartesian form are

$$2x+y+2z+3=0 \text{ and } x-2y+2z-3=0.$$

The vector forms of these equations are

$$\vec{r}\cdot(2\hat{i}+\hat{j}+2\hat{k})+3=0 \text{ and } \vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=3.$$

EXAMPLE 11 *If the points* $(1, 1, p)$ *and* $(-3, 0, 1)$ *be equidistant from the plane* $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$, *find the values of p.*

SOLUTION The equation of the given plane in Cartesian form is

$$x\hat{i}+y\hat{j}+z\hat{k}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$$

$$\Rightarrow 3x+4y-12z+13=0. \quad \text{... (i)}$$

It is being given that the distances of the plane (i) from each of the points $A(1, 1, p)$ and $B(-3, 0, 1)$ are equal.

$$\therefore \quad \frac{|(3\times1)+(4\times1)-(12\times p)+13|}{\sqrt{3^2+4^2+(-12)^2}} = \frac{|(3\times(-3)+(4\times0)-(12\times1)+13|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$|20-12p| = |-8| \Rightarrow |20-12p| = 8$$

$$\Rightarrow (20-12p) = 8 \text{ or } -(20-12p) = 8$$

$$\Rightarrow 12p = 2 \text{ or } 12p = 28$$

$$\Rightarrow p = 1 \text{ or } p = \frac{7}{3}.$$

Hence, $p = 1$ or $p = \frac{7}{3}$.

EXAMPLE 12 *Find the equation of the plane mid-parallel to the planes* $2x - 2y + z + 3 = 0$ *and* $2x - 2y + z + 9 = 0$.

SOLUTION Let the required equation of the plane be $2x - 2y + z + k = 0$. This plane equidistant from each of the given planes. Let $P(x_1, y_1, z_1)$ be any point on the plane $2x - 2y + z + k = 0$.

Then, $2x_1 - 2y_1 + z_1 + k = 0 \Rightarrow 2x_1 - 2y_1 + z_1 = -k$. ... (i)

$\therefore$ $P(x_1, y_1, z_1)$ is equidistant from the planes $2x - 2y + z + 3 = 0$ and $2x - 2y + z + 9 = 0$.

$$\therefore \quad \frac{|2x_1 - 2y_1 + z_1 + 3|}{\sqrt{2^2+(-2)^2+1^2}} = \frac{|2x_1 - 2y_1 + z_1 + 9|}{\sqrt{2^2+(-2)^2+1^2}}$$

$$\Rightarrow |(2x_1 - 2y_1 + z_1) + 3| = |(2x_1 - 2y_1 + z_1) + 9|$$

$$\Rightarrow |-k + 3| = |-k + 9| \text{ [using (i)]}$$

$$\Rightarrow (-k + 3) = (-k + 9) \text{ or } -(-k + 3) = -k + 9$$

$$\Rightarrow k - 3 = -k + 9 \Rightarrow 2k = 12 \Rightarrow k = 6 \text{ [}\because\ (-k + 3) \neq (-k + 9)\text{]}.$$

Hence, the required equation of the plane is $2x - 2y + z + 6 = 0$.

EXAMPLE 13 *A variable plane which remains at a constant distance* $3p$ *from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of* $\triangle ABC$ *is* $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

SOLUTION Let the equation of the variable plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad \ldots \text{(i)}$$

This plane meets the x-axis, y-axis and z-axis at the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ respectively. Let (α, β, γ) be the coordinates of the centroid of $\triangle ABC$.

Then, $\alpha = \frac{a + 0 + 0}{3}$, $\beta = \frac{0 + b + 0}{3}$ and $\gamma = \frac{0 + 0 + c}{3}$

$$\Rightarrow \alpha = \frac{a}{3}, \ \beta = \frac{b}{3} \text{ and } \gamma = \frac{c}{3} \Rightarrow a = 3\alpha, \ b = 3\beta \text{ and } c = 3\gamma. \quad \ldots \text{(ii)}$$

$\therefore$ $3p$ = length of the perpendicular from (0, 0, 0) to the plane (i)

$$\Rightarrow \quad 3p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = \frac{1}{3p}$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{9p^2} \Rightarrow \frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2} = \frac{1}{9p^2} \text{ [using (ii)]}$$

$$\Rightarrow \quad \alpha^{-2}+\beta^{-2}+\gamma^{-2} = p^{-2}.$$

Hence, the required locus is $x^{-2}+y^{-2}+z^{-2} = p^{-2}$.

EXAMPLE 14 *A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is* $x^{-2}+y^{-2}+z^{-2} = 16p^{-2}$.

SOLUTION Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1. \qquad \ldots(i)$$

This plane meets the x-axis, y-axis and z-axis at the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ respectively.

Let (α, β, γ) be the coordinates of the centroid of the tetrahedron $OABC$.

Then, $\alpha = \frac{0+a+0+0}{4} = \frac{a}{4}$, $\beta = \frac{0+0+b+0}{4} = \frac{b}{4}$ and

$$\gamma = \frac{0+0+0+c}{4} = \frac{c}{4} \Rightarrow a = 4\alpha,\ b = 4\beta,\ c = 4\gamma \qquad \ldots \text{(ii)}$$

$\therefore$ p = distance of the plane (i) from (0, 0, 0)

$$\Rightarrow \quad p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \quad \frac{1}{p} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = \frac{1}{p^2} = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2} \text{ [using (ii)]}$$

$$\Rightarrow \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{16}{p^2} \Rightarrow \alpha^{-2}+\beta^{-2}+\gamma^{-2} = 16p^{-2}.$$

Hence, the required locus is $x^{-2}+y^{-2}+z^{-2} = 16p^{-2}$.

EXERCISE 28C

1. Find the distance of the point $(2\hat{i} - \hat{j} - 4\hat{k})$ from the plane $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 9$.
2. Find the distance of the point $(\hat{i} + 2\hat{j} + 5\hat{k})$ from the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + 17 = 0$.
3. Find the distance of the point $(3, 4, 5)$ from the plane $\vec{r} \cdot (2\hat{i} - 5\hat{j} + 3\hat{k}) = 13$.
4. Find the distance of the point $(1, 1, 2)$ from the plane $\vec{r} \cdot (2\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$.
5. Find the distance of the point $(2, 1, 0)$ from the plane $2x + y + 2z + 5 = 0$.
6. Find the distance of the point $(2, 1, -1)$ from the plane $x - 2y + 4z = 9$.
7. Show that the point $(1, 2, 1)$ is equidistant from the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 5$ and $\vec{r} \cdot (2\hat{i} - 2\hat{j} + \hat{k}) + 3 = 0$.
8. Show that the points $(-3, 0, 1)$ and $(1, 1, 1)$ are equidistant from the plane $3x + 4y - 12z + 13 = 0$.
9. Find the distance between the parallel planes $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$.
10. Find the distance between the parallel planes $x + 2y - 2z + 4 = 0$ and $x + 2y - 2z - 8 = 0$.
11. Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$, each one of which is at a unit distance from the point $(1, 1, 1)$.
12. Find the equation of the plane parallel to the plane $2x - 3y + 5z + 7 = 0$ and passing through the point $(3, 4, -1)$. Also, find the distance between the two planes.
13. Find the equation of the plane mid-parallel to the planes $2x - 3y + 6z + 21 = 0$ and $2x - 3y + 6z - 14 = 0$.

ANSWERS (EXERCISE 28C)

1. $\frac{47}{13}$ units **2.** $\frac{25\sqrt{3}}{3}$ units **3.** $\frac{6}{19}\sqrt{38}$ units **4.** $\frac{13}{12}\sqrt{6}$ units

5. $\frac{10}{3}$ units **6.** $\frac{13}{21}\sqrt{21}$ units **9.** $\frac{2}{29}\sqrt{29}$ units **10.** $\frac{5}{14}\sqrt{14}$ units

11. $x - 2y + 2z + 2 = 0$, $x - 2y + 2z - 4 = 0$

12. $2x - 3y + 5z + 11 = 0$, $\frac{2}{19}\sqrt{38}$ units **13.** $4x - 6y + 12z + 7 = 0$

Equation of a Plane Parallel to a Given Plane

VECTOR FORM

Any plane parallel to $\vec{r} \cdot \vec{n} = q_1$ is given by $\vec{r} \cdot \vec{n} = q_2$, where the constant q_2 is determined by a given condition.

CARTESIAN FORM

Let $ax + by + cz + d = 0$ be a given plane.

Then, any plane parallel to this plane is of the form $ax + by + cz + k = 0$, where k is determined by a given condition.

DISTANCE BETWEEN TWO PARALLEL PLANES

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ be any two parallel planes. Then, we take a point $P(x_1, y_1, z_1)$ on any of these planes and find the length of perpendicular drawn from $P(x_1, y_1, z_1)$ to the other plane.

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane which is parallel to the plane* $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$ *and passes through the point whose position vector is* $(\hat{i} + \hat{j} + \hat{k})$.

SOLUTION Any plane parallel to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$ is given by

$$\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = q \text{ for some constant } q. \quad \text{... (i)}$$

Since it passes through a point having position vector $(\hat{i} + \hat{j} + \hat{k})$, we have

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = q \Rightarrow q = (2 - 1 + 2) = 3.$$

Putting $q = 3$ in (i), we get the required equation of the plane as

$$\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 3.$$

EXAMPLE 2 *Find the vector equation of a plane which is parallel to the plane* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ *and passes through the point* $(3, 4, -1)$.

SOLUTION Any plane parallel to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ is given by $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) = q$ for some constant q. ... (i)

Since it passes through a point $A(3, 4, -1)$ whose position vector is $(3\hat{i} + 4\hat{j} - \hat{k})$, we have

$$(3\hat{i}+4\hat{j}-\hat{k})\cdot(2\hat{i}-3\hat{j}+5\hat{k})=q \Rightarrow q=(6-12-5)=-11.$$

Putting $q=-11$ in (i), we get the required equation of the plane as

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})=-11 \Rightarrow \vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0.$$

EXAMPLE 3 *Find the equation of the plane which is parallel to the plane* $2x-3y+z+8=0$ *and which passes through the point* $(-1, 1, 2)$.

SOLUTION Any plane parallel to the given plane is

$$2x-3y+z+k=0 \text{ for some constant } k. \quad \text{... (i)}$$

If it passes through the point $A(-1, 1, 2)$, then

$$2\times(-1)-3\times1+2+k=0$$

$$\Rightarrow (-2-3+2)+k=0 \Rightarrow -3+k=0 \Rightarrow k=3.$$

Hence, the required equation of the plane is $2x-3y+z+3=0$.

EXAMPLE 4 *Find the distance between the planes* $2x-y+3z+4=0$ *and* $6x-3y+9z-3=0$.

SOLUTION Clearly, we have $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

$\therefore$ the given planes are parallel.

Let $P(x_1, y_1, z_1)$ be any point on the plane $2x-y+3z+4=0$.

Then, $2x_1-y_1+3z_1+4=0 \Rightarrow 2x_1-y_1+3z_1=-4$. ... (i)

Let p be the distance of the point $P(x_1, y_1, z_1)$ from the plane $6x-3y+9z-3=0$. Then,

$$p=\frac{|6x_1-3y_1+9z_1-3|}{\sqrt{6^2+(-3)^2+9^2}}=\frac{|3(2x_1-y_1+3z_1)-3|}{\sqrt{36+9+81}}$$

$$=\frac{|3\times(-4)-3|}{\sqrt{126}} \text{ [using (i)]}$$

$$=\frac{|-15|}{3\sqrt{14}}=\frac{15}{3\sqrt{14}}=\frac{5}{\sqrt{14}} \text{ units.}$$

Hence, the distance between the given planes is $\frac{5}{\sqrt{14}}$ units.

EXERCISE 28D

1. Show that the planes $2x-y+6z=5$ and $5x-2.5y+15z=12$ are parallel.
2. Find the vector equation of the plane through the point $(3\hat{i}+4\hat{j}-\hat{k})$ and parallel to the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+5=0$.

3. Find the vector equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$.
4. Find the vector equation of the plane passing through the point (1, 1, 1) and parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$.
5. Find the equation of the plane passing through the point (1, 4, –2) and parallel to the plane $2x-y+3z+7=0$.
6. Find the equation of the plane passing through the origin and parallel to the plane $5x-3y+7z+13=0$.
7. Find the equation of the plane passing through the point (–1, 0, 7) and parallel to the plane $3x-5y+4z=11$.
8. Find the equations of planes parallel to the plane $x-2y+2z=3$ which are at a unit distance from the point (1, 2, 3).
9. Find the distance between the planes $x+2y+3z+7=0$ and $2x+4y+6z+7=0$.

ANSWERS (EXERCISE 28D)

2. $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$ **3.** $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=a+b+c$

4. $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=3$ **5.** $2x-y+3z+8=0$

6. $5x-3y+7z=0$ **7.** $3x-5y+4z=25$

8. $x-2y+2z=0$ or $x-2y+2z-6=0$ **9.** $\frac{7}{\sqrt{56}}$ units

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28D)

1. The given planes are in the form $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$, where $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$. Hence, the given planess are parallel.

8. Let the required plane be $x-2y+2z+k=0$ for some constants k.
Then, its distance from the point $P(1, 2, 3)$ is

$$\frac{|1-2\times2+2\times3+k|}{\sqrt{1^2+(-2)^2+2^2}}=\frac{|3+k|}{3}=1\Rightarrow|3+k|=3$$

$\Rightarrow \quad 3+k=3$ or $3+k=-3\Rightarrow k=0$ or $k=-6$.
Hence, the required equations are $x-2y+2z=0$ or $x-2y+2z-6=0$.

9. Let $P(x_1, y_1 z_1)$ be any point on the plane $x+2y+3z+7=0$.
Then, $x_1+2y_1+3z_1=-7$. ... (i)

$$\therefore \quad p=\frac{|2x_1+4y_1+6z_1+7|}{\sqrt{2^2+4^2+6^2}}=\frac{|2(x_1+2y_1+3z_1)+7|}{\sqrt{56}}$$

$$=\frac{|2\times(-7)+7|}{\sqrt{56}}=\frac{7}{\sqrt{56}}\text{ units.}$$

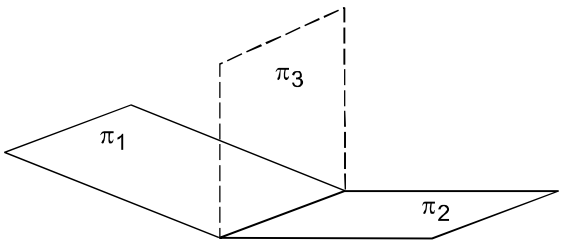
PLANE PASSING THROUGH THE INTERSECTION OF TWO PLANES

THEOREM 1 *The equation of a plane passing through the intersection of two planes* $\vec{r}\cdot\vec{n_1} = q_1$ *and* $\vec{r}\cdot\vec{n_2} = q_2$ *is given by* $\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2}) = (q_1 + \lambda q_2)$.

PROOF Let π_1 and π_2 be the two given planes with equations $\vec{r}\cdot\vec{n_1} = q_1$ and $\vec{r}\cdot\vec{n_2} = q_2$ respectivley.

Let $\vec{t}$ be the position vector of any point on their line of intersection.

Then, it must satisfy both the equations.



$\therefore \quad \vec{t}\cdot\vec{n_1} = q_1$ and $\vec{t}\cdot\vec{n_2} = q_2$

$\Rightarrow \quad \vec{t}\cdot(\vec{n_1}+\lambda\vec{n_2}) = (q_1 + \lambda q_2)$ for every real value λ.

Since $\vec{t}$ is arbitrary, the required equation of a plane π_3 passing through the intersection of the given planes, is given by

$$\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2}) = (q_1 + \lambda q_2).$$

THEOREM 2 *The equation of a plane passing through the intersection of two planes* $a_1x + b_1y + c_1z = d_1$ *and* $a_2x + b_2y + c_2z = d_2$ *is given by*

$$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_z) = 0.$$

PROOF Taking $\vec{n_1} = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k})$, $\vec{n_2} = (a_2\hat{i} + b_2\hat{j} + c_3\hat{k})$, $q_1 = d_1$, $q_2 = d_2$ and $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, we may write the above vector equation as

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot[(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) + \lambda(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = (d_1 + \lambda d_2)]$$

$$\Rightarrow \quad (a_1x + b_1y + c_1z = d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0,$$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

SUMMARY

1. The vector equation of a plane passing through the intersection of two planes $\vec{r}\cdot\vec{n_1} = q_1$ and $\vec{r}\cdot\vec{n_2} = q_2$ is given by
$$\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2}) = (q_1 + \lambda q_2).$$
2. The equation of a plane passing through the intersection of two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by $(a_1x + b_1y + c_1z - d_1 + \lambda(a_2x + b_2y + c_2z - d_2) = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of the plane through the intersection of the planes* $3x - y + 2z = 4$ *and* $x + y + z = 2$ *and passing through the point* $(2, 2, 1)$.

SOLUTION Let the required equation of the plane be

$$(3x - y + 2z - 4) + \lambda(x + y + z - 2) = 0 \text{ for some real value of } \lambda$$

$$\Rightarrow \quad (3 + \lambda)x + (-1 + \lambda)y + (2 + \lambda)z - (4 + 2\lambda) = 0. \qquad \ldots (i)$$

It is given that the point $A(2, 2, 1)$ lies on (i).

$$\therefore \quad (3 + \lambda) \times 2 + (-1 + \lambda) \times 2 + (2 + \lambda) \times 1 - (4 + 2\lambda) = 0$$

$$\Rightarrow \quad (6 + 2\lambda) + (-2 + 2\lambda) + (2 + \lambda) - 4 - 2\lambda = 0$$

$$\Rightarrow \quad 3\lambda = -2 \Rightarrow \lambda = \frac{-2}{3}.$$

Putting $\lambda = \frac{-2}{3}$ in (i), we get

$$\left(3 - \frac{2}{3}\right)x + \left(-1 - \frac{2}{3}\right)y + \left(2 - \frac{2}{3}\right)z - 4 + \frac{4}{3} = 0$$

$$\Rightarrow \quad \frac{7x}{3} - \frac{5y}{3} + \frac{4z}{3} - \frac{8}{3} = 0 \Rightarrow 7x - 5y + 4z - 8 = 0.$$

Hence, the required equation of the plane is $7x - 5y + 4z - 8 = 0$.

EXAMPLE 2 *Find the equation of the plane through the line of intersection of the planes* $x + y + z = 1$ *and* $2x + 3y + 4z = 5$, *which is perpendicular to the plane* $x - y + z = 0$. *Also find the distance of the plane so obtained from the origin.* **[CBSE 2014]**

SOLUTION Let the required equation of the plane be

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

for some real value of λ

$$\Rightarrow \quad (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - (1 + 5\lambda) = 0. \qquad \ldots (i)$$

Since this plane is perpendicular to the plane $x - y + z = 0$, we have

$$(1 + 2\lambda) \times 1 + (1 + 3\lambda) \times (-1) + (1 + 4\lambda) \times 1 = 0$$

$$\Rightarrow \quad (1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) = 0 \Rightarrow 3\lambda = -1 \Rightarrow \lambda = \frac{-1}{3}.$$

Putting $\lambda = \frac{-1}{3}$ in (i), we get

$$\left(1 - \frac{2}{3}\right)x + (1 - 1)y + \left(1 - \frac{4}{3}\right)z - 1 + \frac{5}{3} = 0$$

$$\Rightarrow \quad \frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0 \Rightarrow x - z + 2 = 0.$$

Hence, the required equation of the plane is $x - z + 2 = 0$.

Distance of the plane from the origin

= Length of perpendicular from the origin to the plane

$$= \frac{|0 - 0 + 2|}{\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

EXAMPLE 3 *Find the equation of the plane passing through the line of intersection of the planes* $2x + y - z = 3$ *and* $5x - 3y + 4z + 9 = 0$ *and parallel to the line*

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}.$$ **[CBSE 2011]**

SOLUTION Let the required equation of the plane be

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0$$

$$\Rightarrow \quad (2+5\lambda)x + (1-3\lambda)y + (-1+4\lambda)z - 3 + 9\lambda = 0. \qquad \text{... (i)}$$

The plane given by (i) is parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}. \qquad \text{... (ii)}$$

So, the normal to the plane (i) is perpendicular to the line (ii).

$$\therefore \quad 2(2+5\lambda) + 4(1-3\lambda) + 5(-1+4\lambda) = 0$$

$$\Rightarrow \quad (4+10\lambda) + (4-12\lambda) + (-5+20\lambda) = 0$$

$$\Rightarrow \quad 18\lambda = -3 \Rightarrow \lambda = \frac{-3}{18} \Rightarrow \lambda = \frac{-1}{6}.$$

Putting $\lambda = \frac{-1}{6}$ in (i), we get

$$\left(2 - \frac{5}{6}\right)x + \left(1 + \frac{3}{6}\right)y + \left(-1 - \frac{4}{6}z\right) - 9 \times \frac{1}{6} = 0$$

$$\Rightarrow \quad \frac{7x}{6} + \frac{9y}{6} - \frac{10z}{6} - \frac{27}{6} = 0$$

$$\Rightarrow \quad 7x + 9y - 10z - 27 = 0.$$

Hence, the required equation of the plane is $7x + 9y - 10z - 27 = 0$.

EXAMPLE 4 *Find the equation of the plane passing through the intersection of the planes* $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ *and* $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ *and parallel to the x-axis.* **[CBSE 2011]**

SOLUTION Converting the given equations of the planes in Cartesian form, we get

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$$

$$\Rightarrow \quad x + y + z - 1 = 0, \qquad \text{... (i)}$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$$

$$\Rightarrow \quad 2x + 3y - z + 4 = 0. \qquad \text{... (ii)}$$

Now, the equation of a plane passing through the intersection of the planes (i) and (ii) is given by

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0 \text{ for some real value of } \lambda$$

$$\Rightarrow \quad (1+2\lambda)x + (1+3\lambda)y + (1-\lambda)z + (-1+4\lambda) = 0. \qquad \text{... (iii)}$$

D.r.'s of the normal to the plane are $(1+2\lambda),(1+3\lambda),(1-\lambda)$.

D.r.'s of the x-axis are 1, 0, 0.

Plane (iii) is parallel to the x-axis means that the normal to this plane is perpendicular to the x-axis.

$$\therefore \quad 1\times(1+2\lambda)+0\times(1+3\lambda)+0\times(1-\lambda)=0 \Rightarrow 1+2\lambda=0 \Rightarrow \lambda=\frac{-1}{2}.$$

Putting $\lambda=\frac{-1}{2}$ in (iii), we get the required equation of the plane as

$$(1-1)x+\left(1-\frac{3}{2}\right)y+\left(1+\frac{1}{2}\right)z+(-1-2)=0$$

$$\Rightarrow \quad \frac{-y}{2}+\frac{3z}{2}-3=0 \Rightarrow y-3z+6=0.$$

Hence, the required equation of the plane is $y-3z+6=0$.

EXAMPLE 5 *Find the equation of the plane passing through the line of intersection of the planes* $\vec{r}\cdot(\hat{i}+3\hat{j})-6=0$ *and* $\vec{r}\cdot(3\hat{i}-\hat{j}-4\hat{k})=0$, *whose perpendicular distance from the origin is unity.* **[CBSE 2013]**

SOLUTION Taking $\vec{r}=(x\hat{i}+y\hat{j}+z\hat{k})$, the equations of the given planes are

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+3\hat{j})-6=0 \Rightarrow x+3y-6=0, \quad \text{... (i)}$$

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}-4\hat{k})=0 \Rightarrow 3x-y-4z=0. \quad \text{... (ii)}$$

The equation of any plane passing through the line of intersection of the given planes is given by

$$(x+3y-6)+\lambda(3x-y-4z) \text{ for some real number } \lambda.$$

$$\Rightarrow \quad (1+3\lambda)x+(3-\lambda)y-4\lambda z-6=0. \quad \text{... (iii)}$$

Length of perpendicular from origin to plane (iii) is given as 1.

$$\therefore \quad \frac{|0+0-0-6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}=1$$

$$\Rightarrow \quad (1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2=36$$

$$\Rightarrow \quad 26\lambda^2=26 \Rightarrow \lambda^2=1 \Rightarrow \lambda=\pm 1.$$

Putting $\lambda=1$ in (iii), we get

$$4x+2y-4z-6=0 \Rightarrow 2x+y-2z-3=0.$$

Putting $\lambda=-1$ in (iii), we get

$$-2x+4y+4z-6=0 \Rightarrow x-2y-2z+3=0.$$

Hence, the required equations of the plane are

$$2x+y-2z-3=0 \text{ and } x-2y-2z+3=0.$$

Note: The vector equations of these planes are

$$\vec{r}\cdot(2\hat{i}+\hat{j}-2\hat{k})=3 \text{ and } \vec{r}\cdot(\hat{i}-2\hat{j}-2\hat{k})+3=0.$$

EXAMPLE 6 *Find the vector equation of the plane passing through the intersection of the planes* $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=7$ *and* $\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$ *and passing through the point* (2, 1, 3).

SOLUTION The equations of the given planes are $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=7$ and $\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$.

These are of the form $\vec{r}\cdot\vec{n}_1=q_1$ and $\vec{r}\cdot\vec{n}_2=q_2$, where

$\vec{n}_1=(2\hat{i}+2\hat{j}-3\hat{k}), \vec{n}_2=(2\hat{i}+5\hat{j}+3\hat{k}), q_1=7$ and $q_2=9$.

The equation of the plane through the intersection of the given planes is given by

$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=(q_1+\lambda q_2)$ for some real number λ

$\Rightarrow \quad \vec{r}\cdot[(2\hat{i}+2\hat{j}-3\hat{k})+\lambda(2\hat{i}+5\hat{j}+3\hat{k})](7+9\lambda)$

$\Rightarrow \quad \vec{r}\cdot[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(-3+3\lambda)\hat{k}]=(7+9\lambda).$ ... (i)

If the plane (i) passes through the point $A(2, 1, 3)$, then

$\vec{r}=(2\hat{i}+\hat{j}+3\hat{k})$ must satisfy it.

$\therefore \quad (2\hat{i}+\hat{j}+3\hat{k})\cdot[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(-3+3\lambda)\hat{k}]=(7+9\lambda)$

$\Rightarrow \quad 2(2+2\lambda)+1\cdot(2+5\lambda)+3(-3+3\lambda)=7+9\lambda$

$\Rightarrow \quad (4+4\lambda)+(2+5\lambda)+(-9+9\lambda)=7+9\lambda \Rightarrow 9\lambda=10 \Rightarrow \lambda=\frac{10}{9}.$

Putting $\lambda=\frac{10}{9}$ in (i), we get the required equation of the plane as

$$\vec{r}\cdot\left[\left(2+\frac{20}{9}\right)\hat{i}+\left(2+\frac{50}{9}\right)\hat{j}+\left(-3+\frac{30}{9}\right)\hat{k}\right]=(7+10)$$

$\Rightarrow \quad \vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153.$

Hence, the required equation of the plane is

$\vec{r}\cdot(38\hat{i}+68\hat{j}+3\hat{k})=153.$

EXAMPLE 7 *Find the vector equation of the plane through the line of intersection of the planes* $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$ *and* $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ *and perpendicular to the plane* $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0.$ **[CBSE 2011]**

SOLUTION The equations of the given planes are

$$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=4 \text{ and } \vec{r}\cdot(-2\hat{i}-\hat{j}+\hat{k})=5.$$

These equations are of the form $\vec{r}\cdot\vec{n}_1=q_1$ and $\vec{r}\cdot\vec{n}_2=q_2$, where

$$\vec{n}_1=(\hat{i}+2\hat{j}+3\hat{k}), \vec{n}_2=(-2\hat{i}-\hat{j}+\hat{k}), q_1=4 \text{ and } q_2=5.$$

The equation of the plane through the intersection of the given planes is given by

$$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=(q_1+\lambda q_2) \text{ for some real number } \lambda$$

$$\Rightarrow \quad \vec{r}\cdot[(\hat{i}+2\hat{j}+3\hat{k})+\lambda(-2\hat{i}-\hat{j}+\hat{k})]=(4+5\lambda)$$

$$\Rightarrow \quad \vec{r}\cdot[(1-2\lambda)\hat{i}+(2-\lambda)\hat{j}+(3+\lambda)\hat{k}]=(4+5\lambda) \qquad \text{... (i)}$$

This plane (i) is perpendicular to the plane $\vec{r}\cdot(-5\hat{i}-3\hat{j}+6\hat{k})=8$, so we have

$$(1-2\lambda)\times(-5)+(2-\lambda)\times(-3)+(3+\lambda)\times 6=0$$

$$\Rightarrow \quad (-5+10\lambda)+(-6+3\lambda)+(18+6\lambda)=0$$

$$\Rightarrow \quad 19\lambda=-7 \Rightarrow \lambda=\frac{-7}{19}.$$

Putting $\lambda=\frac{-7}{19}$ in (i), we get

$$\vec{r}\cdot\left[\left(1+\frac{14}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right]=\left(4-\frac{35}{19}\right)$$

$$\Rightarrow \quad \vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})=41.$$

Hence, the required equation of the plane is

$$\vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})=41.$$

EXERCISE 28E

1. Find the equation of the plane through the line of intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$, and passing through the point (1, 1, 1).
2. Find the equation of the plane through the line of intersection of the planes $x-3y+z+6=0$ and $x+2y+3z+5=0$, and passing through the origin.
3. Find the equation of the plane passing through the intersection of the planes $2x+3y-z+1=0$ and $x+y-2z+3=0$, and perpendicular to the plane $3x-y-2z-4=0$.
4. Find the equation of the plane passing through the line of intersection of the planes $2x-y=0$ and $3z-y=0$, and perpendicular to the plane $4x+5y-3z=9$.

5. Find the equation of the plane passing through the intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 8$, and parallel to the line with direction ratios 1, 2, 1. Also, find the perpendicular distance of (1, 1, 1) from the plane. **[CBSE 2005]**

6. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z - 5 = 0$ and $3x - 2y - z + 1 = 0$ and cutting off equal intercepts on the x-axis and z-axis.

7. Find the equation of the plane through the intersection of the planes $3x - 4y + 5z = 10$ and $2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

   HINT: The equations of the given line are $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$.

8. Find the vector equation of the plane through the intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 0$ and $\vec{r} \cdot (\hat{j} + 2\hat{k}) = 0$, and passing through the point (2, 1, –1).

9. Find the vector equation of the plane through the point (1, 1, 1), and passing through the intersection of the planes
   $$\vec{r} \cdot (\hat{i} - \hat{j} + 3\hat{k}) + 1 = 0 \text{ and } \vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0.$$

10. Find the vector equation of the plane passing through the intersection of the planes
    $$\vec{r} \cdot (2\hat{i} - 7\hat{j} + 4\hat{k}) = 3 \text{ and } \vec{r} \cdot (3\hat{i} - 5\hat{j} + 4\hat{k}) + 11 = 0,$$
    and passing through the point (–2, 1, 3). **[CBSE 2006]**

11. Find the equation of the plane through the line of intersection of the planes $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$.

12. Find the Cartesian and vector equations of the planes through the line of intersection of the planes $\vec{r} \cdot (\hat{i} - \hat{j}) + 6 = 0$ and $\vec{r} \cdot (3\hat{i} + 3\hat{j} - 4\hat{k}) = 0$, which are at a unit distance from the origin.

ANSWERS (EXERCISE 28E)

1. $20x + 23y + 26z = 69$
2. $x + 27y + 13z = 0$
3. $7x + 13y + 4z = 9$
4. $28x - 17y + 9z = 0$
5. $9x - 8y + 7z = 21$, $\frac{13}{\sqrt{194}}$ units
6. $5x + 2y + 5z - 9 = 0$
7. $x - 20y + 27z = 14$
8. $\vec{r} \cdot (\hat{i} + 9\hat{j} + 11\hat{k}) = 0$
9. $\vec{r} \cdot (11\hat{i} + \hat{j} + 5\hat{k}) - 17 = 0$
10. $\vec{r} \cdot (15\hat{i} - 47\hat{j} + 28\hat{k}) = 7$
11. $\vec{r} \cdot (-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47$

12. $2x + y - 2z + 3 = 0,\ x + 2y - 2z - 3 = 0$, and

$\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) + 3 = 0,\ \vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) - 3 = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28E)

5. Let the required plane be

$$(x - 2y + z - 1) + \lambda(2x + y + z - 8) = 0$$

$$\Leftrightarrow \quad (1 + 2\lambda)x + (\lambda - 2)y + (1 + \lambda)z - (1 + 8\lambda) = 0. \qquad \ldots \text{(i)}$$

The d.r.'s of the normal to this plane are

$$(1 + 2\lambda), (\lambda - 2), (1 + \lambda).$$

The normal to the plane (i) is perpendicular to the line with d.r.'s 1, 2, 1.

$$\therefore \quad (1 + 2\lambda) + 2(\lambda - 2) + (1 + \lambda) = 0 \Rightarrow \lambda = \frac{2}{5}.$$

6. Let the required plane be

$$(x + 2y + 3z - 5) + \lambda(3x - 2y - z + 1) = 0.$$

Then,

$$(1 + 3\lambda)x + (2 - 2\lambda)y + (3 - \lambda)z - 5 + \lambda = 0.$$

$$(1 + 3\lambda)x + (2 - 2\lambda)y + (3 - \lambda)z = 5 - \lambda$$

$$\Rightarrow \quad \frac{(1 + 3\lambda)}{(5 - \lambda)}x + \frac{(2 - 2\lambda)}{(5 - \lambda)}y + \frac{(3 - \lambda)}{(5 - \lambda)}z = 1$$

$$\Rightarrow \quad \frac{x}{\left(\frac{5 - \lambda}{1 + 3\lambda}\right)} + \frac{y}{\left(\frac{5 - \lambda}{2 - 2\lambda}\right)} + \frac{z}{\left(\frac{5 - \lambda}{3 - \lambda}\right)} = 1.$$

Since the intercepts on the x-axis and z-axis are equal, we have

$$\frac{5 - \lambda}{1 + 3\lambda} = \frac{5 - \lambda}{3 - \lambda} \Rightarrow 3 - \lambda = 1 + 3\lambda$$

$$\Rightarrow 4\lambda = 2 \Rightarrow \lambda = \frac{1}{2}.$$

7. Let the required plane be

$$(3x - 4y + 5z - 10) + \lambda(2x + 2y - 3z - 4) = 0$$

$$\Rightarrow \quad (3 + 2\lambda)x + (2\lambda - 4)y + (5 - 3\lambda)z - 10 - 4\lambda = 0. \qquad \ldots \text{(i)}$$

D.r's of normal to the plane are $(3 + 2\lambda), (2\lambda - 4), (5 - 3\lambda)$.

Given line is $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$.

D.r.'s of the line are 6, 3, 2.

$\therefore$ This line is perpendicular to the plane.

$$\therefore \quad 6(3 + 2\lambda) + 3(2\lambda - 4) + 2(5 - 3\lambda) = 0 \Rightarrow 12\lambda = -16 \Rightarrow \lambda = \frac{-4}{3}.$$

---

12. The equations of the given planes are

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-\hat{j})+6=0 \text{ and } (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}+3\hat{j}-4\hat{k})=0$$

$\Rightarrow \quad x-y+6=0$ and $3x+3y-4z=0.$

Any plane through their intersection is

$$(x-y+6)+\lambda(3x+3y-4z)=0 \Rightarrow (1+3\lambda)x+(3\lambda-1)y-4\lambda z+6=0. \quad \ldots \text{(i)}$$

$$\therefore \quad \frac{6}{\sqrt{(1+3\lambda)^2+(3\lambda-1)^2+(-4\lambda)^2}}=1 \Rightarrow 34\lambda^2+2=36 \Rightarrow \lambda^2=1 \Rightarrow \lambda=\pm 1$$

So, the required planes are

$2x+y-2z+3=0$ and $x+2y-2z-3=0.$

In vector form, they are

$\vec{r}\cdot(2\hat{i}+\hat{j}-2\hat{k})+3=0$ and $\vec{r}\cdot(\hat{i}+2\hat{j}-2\hat{k})-3=0.$

Equation of a Plane Passing through Three Noncollinear Points

Vector Form

THEOREM 1 *The vector equation of a plane passing through three noncollinear points with position vectors* $\vec{a}, \vec{b}, \vec{c}$ *is*

$$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0.$$

PROOF Let A, B, C be three given noncollinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively.

Let P be an arbitrary point on the plane passing through the points A, B, C, and let $\vec{r}$ be the position vector of P. Then,

$$\overrightarrow{AP}=(\text{p.v. of } P)-(\text{p.v. of } A)=(\vec{r}-\vec{a}),$$

$$\overrightarrow{AB}=(\text{p.v. of } B)-(\text{p.v. of } A)=(\vec{b}-\vec{a}),$$

$$\overrightarrow{AC}=(\text{p.v. of } C)-(\text{p.v. of } A)=(\vec{c}-\vec{a}).$$

Since the points A, B, C, P lie on the plane, the vectors $\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar.

So, the scalar triple product of these vectors is 0.

$$\therefore \quad [\overrightarrow{AP}\ \overrightarrow{AB}\ \overrightarrow{AC}]=0$$

$$\Rightarrow \quad \overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$$

$$\Rightarrow \quad (\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0,$$

which is the required equation of the plane.

Cartesian Form

THEOREM 2 *The equation of a plane passing through three given noncollinear points* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)$ *is given by*

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

PROOF Let $P(x, y, z)$ be an arbitrary point on the given plane. Then,

$$\begin{aligned} \overrightarrow{AP} &= (\text{p.v. of } P) - (\text{p.v. of } A) \\ &= (x\hat{i} + y\hat{j} + z\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}. \end{aligned}$$

Similarly, we have

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

and $\overrightarrow{AC} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}.$

Since $\overrightarrow{AP}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar, we have

$$[\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}] = 0.$$

Hence, the required equation of the plane passing through three given points A, B, C is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

SUMMARY

(i) The vector equation of a plane passing through three noncollinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is given by

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0.$$

(ii) The Cartesian equation of a plane passing through three noncollinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane passing through the points* $A(2, 5, -3)$, $B(-2, -3, 5)$ *and* $C(5, 3, -3)$.

SOLUTION The position vectors of the given points A, B, and C are $\vec{a} = (2\hat{i} + 5\hat{j} - 3\hat{k})$, $\vec{b} = (-2\hat{i} - 3\hat{j} + 5\hat{k})$ and $\vec{c} = (5\hat{i} + 3\hat{j} - 3\hat{k})$ respectively.

Then, the vector equation of the required plane is

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0.$$

Now $(\vec{b} - \vec{a}) = (-2 - 2)\hat{i} + (-3 - 5)\hat{j} + (5 + 3)\hat{k} = (-4\hat{i} - 8\hat{j} + 8\hat{k})$.

and $(\vec{c} - \vec{a}) = (5 - 2)\hat{i} + (3 - 5)\hat{j} + (-3 + 3)\hat{k} = (3\hat{i} - 2\hat{j} + 0\hat{k})$.

$$\therefore \quad (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$

$$= (0 + 16)\hat{i} - (0 - 24)\hat{j} + (8 + 24)\hat{k}$$

$$= (16\hat{i} + 24\hat{j} + 32\hat{k}).$$

$\therefore$ the required vector equation is

$$(\vec{r} - \vec{a}) \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 0$$

$$\Rightarrow \quad \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = \vec{a} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= (32 + 120 - 96) = 56.$$

Hence, the required vector equation of the plane is

$$\vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 56 \Rightarrow \vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7.$$

EXAMPLE 2 *Find the equation of the plane passing through the points* $A(1, 1, 0)$, $B(1, 2, 1)$ *and* $C(-2, 2, -1)$.

SOLUTION Here

$$(x_1 = 1, y_1 = 1, z_1 = 0), (x_2 = 1, y_2 = 2, z_2 = 1)$$

and $(x_3 = -2, y_3 = 2, z_3 = -1)$.

Hence, the required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-0 \\ 1-1 & 2-1 & 1-0 \\ -2-1 & 2-1 & -1-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(-1-1)-(y-1)(0+3)+z(0+3)=0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z=0$$

$$\Rightarrow -2x-3y+3z+5=0 \Rightarrow 2x+3y-3z-5=0.$$

Hence, the required equation of the plane is $2x+3y-3z-5=0$.

ANGLE BETWEEN TWO PLANES

The angle between two given planes is the angle between their normals.

If θ is an angle between two planes, then $(180^\circ - \theta)$ is also an angle between them.

An Important Note: We shall take the acute angle as the angle between two planes.

Vector Form

Let θ be the angle between two planes whose vector equations are

$$\vec{r}\cdot\vec{n}_1 = q_1 \text{ and } \vec{r}\cdot\vec{n}_2 = q_2.$$

Then, θ is the angle between their normals $\vec{n}_1$ and $\vec{n}_2$.

$$\therefore \quad \cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}.$$

Two Important Results

1. *Condition for two planes to be perpendicular to each other*

   Two planes $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$ are perpendicular to each other

   $$\Leftrightarrow \vec{n}_1 \perp \vec{n}_2 \Leftrightarrow \vec{n}_1\cdot\vec{n}_2 = 0.$$

2. *Condition for two planes to be parallel to each other*

   Two planes $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$ are parallel to each other

$\Leftrightarrow \vec{n_1} \,||\, \vec{n_2}$

$\Leftrightarrow \vec{n_1} = \lambda \vec{n_2}$ for some scalar λ.

REMARK 1 Any plane parallel to $\vec{r} \cdot \vec{n} = q$ is $\vec{r} \cdot \vec{n} = q_1$.

REMARK 2 The equation of the plane parallel to the plane $\vec{r} \cdot \vec{n} = q$ and passing through the point with position vector $\vec{a}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

Cartesian Form

Let θ be the acute angle between the planes:

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

Then, θ is the angle between their normals.

D.r.'s of the normals to the given planes are

a_1, b_1, c_1 and a_2, b_2, c_2.

$$\therefore \quad \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}.$$

Two Important Results

1. *Condition for two planes to be perpendicular to each other*
   Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other
   $\Leftrightarrow$ their normals are perpendicular to each other
   $\Leftrightarrow$ lines with direction ratios a_1, b_1, c_1, and a_2, b_2, c_2 are perpendicular to each other
   $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.
2. *Condition for two planes to be parallel to each other*
   Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel to each other
   $\Leftrightarrow$ their normals are parallel to each other
   $\Leftrightarrow$ lines with direction ratios a_1, b_1, c_1, and a_2, b_2, c_2 are parallel to each other
   $\Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

REMARK 1 The equation of any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

REMARK 2 The equation of a plane passing through (x_1, y_1, z_1) and parallel to the plane $ax + by + cz + d = 0$ is given by
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

REMARK 3 The equation of any plane parallel to the xy-plane is $z = \lambda$.
This plane is perpendicular to the z-axis.
The equation of any plane parallel to the xz-plane is $y = \lambda$.

xz-plane is perpendiuclar to the y-axis.

The equation of any plane parallel to the yz-plane is $x = \lambda$.

yz-plane is perpendicular to the x-axis.

SUMMARY OF THE RESULTS

1. The acute angle θ between the planes $\vec{r} \cdot \vec{n}_1 = q_1$ and $\vec{r} \cdot \vec{n}_2 = q_2$ is given by

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}.$$

   (i) Two planes $\vec{r} \cdot \vec{n}_1 = q_1$ and $\vec{r} \cdot \vec{n}_2 = q_2$ are perpendicular to each other $\Leftrightarrow \vec{n}_1 \cdot \vec{n}_2 = 0$.

   (ii) Two planes $\vec{r} \cdot \vec{n}_1 = q_1$ and $\vec{r} \cdot \vec{n}_2 = q_2$ are perpendicular to each other $\Leftrightarrow \vec{n}_1 = \lambda \vec{n}_2$ for some scalar λ.

   (iii) Any plane parallel to the plane $\vec{r} \cdot \vec{n} = q$ is $\vec{r} \cdot \vec{n} = q_1$.

   (iv) Any plane parallel to $\vec{r} \cdot \vec{n} = q$ and passing through a point with p.v. $\vec{a}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

2. The acute angle θ between the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0$$

   is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\,\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}.$$

   (i) Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.

   (ii) Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

   (iii) The equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

   (iv) The equation of a plane passing through the point (x_1, y_1, z_1) and parallel to the plane $ax + by + cz + d = 0$ is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

   (v) Any plane parallel to the yz-plane is $x = \lambda$.

   Any plane parallel to the xz-plane is $y = \lambda$.

   Any plane parallel to the xy-plane is $z = \lambda$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the angle between the planes* $\vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 5$ *and* $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 8$.

SOLUTION We know that the angle between the planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ is given by

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}.$$

Here, $\vec{n_1} = (\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{n_2} = (2\hat{i} - \hat{j} + \hat{k})$.

$$\therefore \quad \cos\theta = \frac{|(\hat{i} + \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})|}{|\hat{i} + \hat{j} + 2\hat{k}||2\hat{i} - \hat{j} + \hat{k}|}$$

$$= \frac{|1 \times 2 + 1 \times (-1) + 2 \times 1|}{\{\sqrt{1^2 + 1^2 + 2^2}\}\{\sqrt{2^2 + (-1)^2 + 1^2}\}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}.$$

Hence, the angle between the given planes is $\left(\frac{\pi}{3}\right)$.

EXAMPLE 2 *Find the angle between the planes whose vector equations are* $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ *and* $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$.

SOLUTION We know that the acute angle θ between the planes $\vec{r} \cdot \vec{n_1} = q_1$ and $\vec{r} \cdot \vec{n_2} = q_2$ is given by

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}.$$

Here, $\vec{n_1} = (2\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{n_2} = (3\hat{i} - 3\hat{j} + 5\hat{k})$.

$$\therefore \quad \cos\theta = \frac{|2\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 3\hat{j} + 5\hat{k})|}{|2\hat{i} + 2\hat{j} - 3\hat{k}||3\hat{i} - 3\hat{j} + 5\hat{k}|}$$

$$= \frac{|2 \times 3 + 2 \times (-3) + (-3) \times 5|}{\{\sqrt{2^2 + 2^2 + (-3)^2}\}\{\sqrt{3^2 + (-3)^2 + 5^2}\}}$$

$$= \frac{|-15|}{(\sqrt{17})(\sqrt{43})} = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{15}{\sqrt{731}}\right).$$

Hence, the angle between the given planes is $\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$.

EXAMPLE 3 *Find the value of λ for which the planes*
$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 13$ *and* $\vec{r}\cdot(\lambda\hat{i}+2\hat{j}-7\hat{k}) = 9$
are perpendicular to each other.

SOLUTION We know that the planes $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$ are perpendicular to each other only when $\vec{n}_1\cdot\vec{n}_2 = 0$.

Here, $\vec{n}_1 = (\hat{i}+2\hat{j}+3\hat{k})$ and $\vec{n}_2 = (\lambda\hat{i}+2\hat{j}-7\hat{k})$.

$\therefore$ the given planes are perpendicular to each other

$\Leftrightarrow \quad \vec{n}_1\cdot\vec{n}_2 = 0$

$\Leftrightarrow \quad (\hat{i}+2\hat{j}+3\hat{k})\cdot(\lambda\hat{i}+2\hat{j}-7\hat{k}) = 0$

$\Leftrightarrow \quad 1\times\lambda+2\times2+3\times(-7) = 0 \quad \Leftrightarrow \quad \lambda = 17.$

Hence, the required value of λ is 17.

EXAMPLE 4 *Find the angle between the planes whose Cartesian equations are* $7x+5y+6z+30 = 0$ *and* $3x-y-10z+4 = 0$.

SOLUTION We know that the acute angle between the planes $a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x+b_2y+c_2z+d_2 = 0$ is given by

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\{\sqrt{a_1^2+b_1^2+c_1^2}\}\ \{\sqrt{a_2^2+b_2^2+c_2^2}\}}.$$

Here $a_1 = 7$, $b_1 = 5$, $c_1 = 6$ and $a_2 = 3$, $b_2 = -1$. $c_2 = -10$.

$$\therefore \quad \cos\theta = \frac{|7\times3+5\times(-1)+6(-10)|}{\{\sqrt{7^2+5^2=6^2}\}\ \{\sqrt{3^2+(-1)^2+(-10)^2}\}}$$

$$= \frac{|21-5-60|}{\{\sqrt{49+25+36}\}\{\sqrt{9+1+100}\}}$$

$$= \frac{44}{\{\sqrt{110}\times\sqrt{110}\}} = \frac{44}{110} = \frac{2}{5}.$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{5}\right).$$

Hence, the acute angle between the given planes is $\cos^{-1}\left(\frac{2}{5}\right)$.

EXAMPLE 5 *Find the value of* λ *for which the planes* $2x - 4y + 3z = 7$ *and* $x + 2y + \lambda z = 18$ *are perpendicular to each other.*

SOLUTION The equations of the given planes are

$$2x - 4y + 3z - 7 = 0 \text{ and } x + 2y + \lambda z - 18 = 0.$$

They are of the form

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0$$

where $a_1 = 2, b_1 = -4, c_1 = 3, d_1 = -7$, and

$$a_2 = 1, b_2 = 2, c_2 = \lambda, d_2 = -18.$$

The given planes are perpendicular to each other

$\Leftrightarrow \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Leftrightarrow \quad 2 \times 1 + (-4) \times 2 + 3 \times \lambda = 0$

$\Leftrightarrow \quad 3\lambda = 6 \Leftrightarrow \lambda = 2.$

Hence, the required value of λ is 2.

EXAMPLE 6 *Find the equation of the plane passing through the point* $(-1, 3, 2)$ *and perpendicular to each of the planes* $x + 2y + 3z = 5$ *and* $3x + 3y + z = 0$. **[CBSE 2009, '11]**

SOLUTION Any plane passing through the point $(-1, 3, 2)$ is given by

$$a(x + 1) + b(y - 3) + c(z - 2) = 0. \quad \text{... (i)}$$

Now, (i) being perpendicular to each of the planes

$$x + 2y + 3z - 5 = 0 \text{ and } 3x + 3y + z = 0,$$

we have,

$$(a \times 1) + (b \times 2) + (c \times 3) = 0 \Rightarrow a + 2b + 3c = 0 \quad \text{... (ii)}$$

$$(a \times 3) + (b \times 3) + (c \times 1) = 0 \Rightarrow 3a + 3b + c = 0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(2-9)} = \frac{b}{(9-1)} = \frac{c}{(3-6)}$$

$$\Rightarrow \quad \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} \Rightarrow \frac{a}{7} = \frac{b}{-8} = \frac{c}{3} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = 7\lambda, b = -8\lambda \text{ and } c = 3\lambda.$$

Putting $a = 7\lambda, b = -8\lambda$ and $c = 3\lambda$ in (i), we get

$$7\lambda(x + 1) - 8\lambda(y - 3) + 3\lambda(z - 2) = 0$$

$$\Rightarrow \quad 7(x + 1) - 8(y - 3) + 3(z - 2) = 0$$

$$\Rightarrow \quad 7x - 8y + 3z + 25 = 0,$$

which is the required equation of the plane.

EXAMPLE 7 *Find the vector equation of the plane through the points* $(2, 1, -1)$ *and* $(-1, 3, 4)$ *and perpendicular to the plane* $x - 2y + 4z = 10$. **[CBSE 2013]**

SOLUTION The general equation of a plane passing through the point $A(2, 1, -1)$ is given by

$$a(x-2)+b(y-1)+c(z+1)=0. \quad \text{... (i)}$$

If the point $B(-1, 3, 4)$ lies on plane (i), then we have

$$a(-1-2)+b(3-1)+c(4+1)=0$$

$$\Rightarrow \quad -3a+2b+5c=0. \quad \text{... (ii)}$$

If the plane (i) is perpendicular to the plane $x-2y+4z=10$, then we have

$$(1\times a)-(2\times b)+(4\times c)=0$$

$$\Rightarrow \quad a-2b+4c=0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{a}{(8+10)}=\frac{b}{(5+12)}=\frac{c}{(6-2)}$$

$$\Rightarrow \quad \frac{a}{18}=\frac{b}{17}=\frac{c}{4}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=18\lambda,\ b=17\lambda \text{ and } c=4\lambda.$$

Substituting these values of a, b and c in (i), we get

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$$\Rightarrow \quad 18(x-2)+17(y-1)+4(z+1)=0$$

$$\Rightarrow \quad 18x+17y+4z=49.$$

Required equation of the plane in vector form is given by

$$\vec{r}\cdot(18\hat{i}+17\hat{j}+4\hat{k})=49.$$

EXAMPLE 8 *Find the equation of the plane passing through the point* (1, 3, 2) *and parallel to the plane* $3x-2y+2z+33=0$.

SOLUTION The equation of a plane parallel to the given plane is of the form

$$3x-2y+2z=k \text{ for some scalar } k. \quad \text{... (i)}$$

Since it passes through the point $A(1, 3, 2)$, we have

$$(3\times 1)-(2\times 3)+(2\times 2)=k \ \Rightarrow \ k=(3-6+4)=1.$$

Hence, the required equation of the plane is

$$3x-2y+2z-1=0.$$

EXERCISE 28F

1. Find the acute angle between the following planes:

(i) $\vec{r}\cdot(\hat{i}+\hat{j}-2\hat{k})=5$ and $\vec{r}\cdot(2\hat{i}+2\hat{j}-\hat{k})=9$

(ii) $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=6$ and $\vec{r}\cdot(2\hat{i}-\hat{j}-\hat{k})+3=0$

(iii) $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(-\hat{i}+\hat{j})=4$

(iv) $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=8$ and $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+7=0$

2. Show that the following planes are at right angles:
   (i) $\vec{r}\cdot(4\hat{i}-7\hat{j}-8\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}-4\hat{j}+5\hat{k})+10=0$
   (ii) $\vec{r}\cdot(2\hat{i}+6\hat{j}+6\hat{k})=13$ and $\vec{r}\cdot(3\hat{i}+4\hat{j}-5\hat{k})+7=0$
3. Find the value of λ for which the given planes are perpendicular to each other:
   (i) $\vec{r}\cdot(2\hat{i}-\hat{j}-\lambda\hat{k})=7$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+2\hat{k})=9$
   (ii) $\vec{r}\cdot(\lambda\hat{i}+2\hat{j}+3\hat{k})=5$ and $\vec{r}\cdot(\hat{i}+2\hat{j}-7\hat{k})+11=0$
4. Find the acute angle between the following planes:
   (i) $2x-y+z=5$ and $x+y+2z=7$
   (ii) $x+2y+2z=3$ and $2x-3y+6z=8$
   (iii) $x+y-z=4$ and $x+2y+z=9$
   (iv) $x+y-2z=6$ and $2x-2y+z=11$
5. Show that each of the following pairs of planes are at right angles:
   (i) $3x+4y-5z=7$ and $2x+6y+6z+7=0$
   (ii) $x-2y+4z=10$ and $18x+17y+4z=49$
6. Prove that the plane $2x+3y-4z=9$ is perpendicular to each of the planes $x+2y+2z-7=0$ and $5x+6y+7z=23$.
7. Show that the planes $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$ are parallel.
8. Find the value of λ for which the planes $x-4y+\lambda z+3=0$ and $2x+2y+3z=5$ are perpendicular to each other.
9. Write the equation of the plane passing through the origin and parallel to the plane $5x-3y+7z+11=0$.
10. Find the equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$..
11. Find the equation of the plane passing through the point (1, –2, 7) and parallel to the plane $5x+4y-11z=6$.
12. Find the equation of the plane passing through the point $A(-1, -1, 2)$ and perpendicular to each of the planes $3x+2y-3z=1$ and $5x-4y+z=5$.
    [CBSE 2008]
13. Find the equation of the plane passing through the origin and perpendicular to each of the planes $x+2y-z=1$ and $3x-4y+z=5$. **[CBSE 2004]**
14. Find the equation of the plane that contains the point $A(1, -1, 2)$ and is perpendicular to both the planes $2x+3y-2z=5$ and $x+2y-3z=8$. Hence, find the distance of the point $P(-2, 5, 5)$ from the plane obtained above. **[CBSE 2014]**

15. Find the equation of the plane passing through the points $A(1, -1, 2)$ and $B(2, -2, 2)$ and perpendicular to the plane $6x - 2y + 2z = 9$. **[CBSE 2005]**

16. Find the equation of the plane passing through the points $A(-1, 1, 1)$ and $B(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$.

17. Find the equation of the plane through the points $A(3, 4, 2)$ and $B(7, 0, 6)$ and perpendicular to the plane $2x - 5y = 15$. **[CBSE 2012]**

HINT: The given plane is $2x - 5y + 0z = 15$.

18. Find the equation of the plane through the points $A(2, 1, -1)$ and $B(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$. Also, show that the plane thus obtained contains the line

$$\vec{r} = (-\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k}).$$ **[CBSE 2012]**

ANSWERS (EXERCISE 28F)

1. (i) $\cos^{-1}\left(\frac{\sqrt{6}}{3}\right)$ (ii) $\cos^{-1}\left(\frac{1}{6}\right)$ (iii) $\cos^{-1}\left(\frac{5}{\sqrt{58}}\right)$ (iv) $\cos^{-1}\left(\frac{6}{7}\right)$

3. (i) $\lambda = -2$ (ii) $\lambda = 17$

4. (i) $\frac{\pi}{3}$ (ii) $\cos^{-1}\left(\frac{8}{21}\right)$ (iii) $\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (iv) $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$ **8.** $\lambda = 2$

9. $5x - 3y + 7z = 0$ **10.** $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$ **11.** $5x + 4y - 11z + 80 = 0$

12. $5x + 9y + 11z = 8$ **13.** $x + 2y + 5z = 0$ **14.** $5x - 4y - z - 7 = 0, \sqrt{42}$ units

15. $x + y - 2z + 4 = 0$ **16.** $2x + 2y - 3z + 3 = 0$ **17.** $5x + 2y - 3z - 17 = 0$

18. $18x + 17y + 4z = 49$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28F)

10. We know that any plane parallel to $\vec{r} \cdot \vec{n} = q$ is $\vec{r} \cdot \vec{n} = q_1$.

Let the required plane parallel to the given plane be $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = q_1$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = q_1$

$\Rightarrow (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = q_1$

$\Rightarrow q_1 = (a + b + c)$.

Hence, the required equation is $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = (a + b + c)$.

14. Any plane through $A(1, -1, 2)$ is given by

$$a(x - 1) + b(y + 1) + c(z - 2) = 0. \quad \text{... (i)}$$

Since it is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$, we have

$2a + 3b - 2c = 0$ ... (ii)

$a + 2b - 3c = 0.$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{a}{(-9+4)} = \frac{b}{(-2+6)} = \frac{c}{(4-3)} = \lambda \Rightarrow a = -5\lambda,\ b = 4\lambda,\ c = \lambda.$$

Putting these value in (i), we get the required equation as

$-5\lambda(x-1) + 4\lambda(y+1) + \lambda(z-2) = 0$

$\Rightarrow\ 5(x-1) - 4(y+1) - (z-2) = 0 \Rightarrow 5x - 4y - z - 7 = 0.$

Distance of the point $P(-2, 5, 5)$ from this plane is given by

$$d = \frac{|5\times(-2) - 4\times5 - 5 - 7|}{\sqrt{5^2 + (-4)^2 + (-1)^2}} = \frac{|-42|}{\sqrt{42}} = \frac{42}{\sqrt{42}} = \sqrt{42} \text{ units.}$$

18. Obtain the required equation of the plane as $18x + 17y + 4z = 49$. ... (A)

The given line is $\vec{r} = (3\lambda - 1)\hat{i} + (3 - 2\lambda)\hat{j} + (4 - 5\lambda)\hat{k}$.

Coordinates of any point on this line are $(3\lambda - 1,\ 3 - 2\lambda,\ 4 - 5\lambda)$.

These coordinates satisfy (A), as shown below:

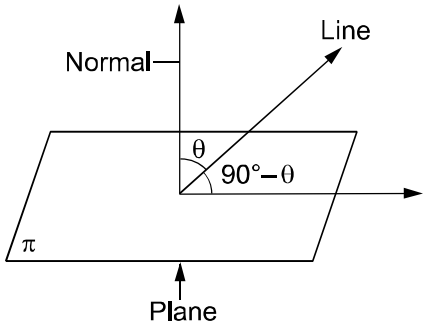
LHS $= 18(3\lambda - 1) + 17(3 - 2\lambda) + 4(4 - 5\lambda) = 49 =$ RHS.

So, $(3\lambda - 1,\ 3 - 2\lambda,\ 4 - 5\lambda)$ lies on plane (A).

Hence, the plane (A) obtained above contains the given line.

ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane.



Vector Form

THEOREM 1 *If ϕ is the angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = q$, then prove that*

$$\sin\phi = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}.$$

PROOF We know that the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to $\vec{b}$.

And, the plane $\vec{r}\cdot\vec{n} = q$ is normal to $\vec{n}$.

Let θ be the angle between the line and the normal to the plane. Then,

$$\cos\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}.$$

Let ϕ be the angle between the line and the plane

Then, $\phi = 90° - \theta \Rightarrow \theta = 90° - \phi$.

$$\therefore \quad \cos(90° - \phi) = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|} \Rightarrow \sin\phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}.$$

Two Important Results

1. *Condition for a given line to be perpendicular to a given plane*

   The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is perpendicular to the plane $\vec{r} \cdot \vec{n} = q$

   $\Leftrightarrow$ $\vec{b}$ is perpendicular to the plane $\vec{r} \cdot \vec{n} = q$

   $\Leftrightarrow$ $\vec{b}$ is parallel to the normal $\vec{n}$ to the plane

   $\Leftrightarrow$ $\vec{b} = t\vec{n}$, for some scalar t.

2. *Condition for a given line to be parallel to a given plane*

   The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$

   $\Leftrightarrow$ $\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$

   $\Leftrightarrow$ $\vec{b}$ is perpendicular to the normal $\vec{n}$ to the plane

   $\Leftrightarrow$ $\vec{b} \cdot \vec{n} = 0$.

Cartesian Form

THEOREM 2 *If ϕ is the angle between the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z + d = 0$, then*

$$\sin\phi = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left\{\sqrt{a_1^2 + b_1^2 + c_1^2}\right\}\left\{\sqrt{a_2^2 + b_2^2 + c_2^2}\right\}}.$$

PROOF The direction ratios of the given line are a_1, b_1, c_1.

So, the given line is parallel to $\vec{b} = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k})$.

The normal to the given plane is parallel to $\vec{n} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Let ϕ be the angle between the line and the plane. Then,

$$\sin\phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}$$

$$= \frac{|(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k})|}{|(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}|\ |(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}|}$$

$$= \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left\{\sqrt{a_1^2 + b_1^2 + c_1^2}\right\}\left\{\sqrt{a_2^2 + b_2^2 + c_2^2}\right\}}.$$

Important Results

1. *Condition for the given line to be perpendicular to the given plane*

   The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is perpendicular to the plane

   $a_2x + b_2y + c_2z + d = 0$

   $\Leftrightarrow$ the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is parallel to the normal to the plane

   $a_2x + b_2y + c_2z + d = 0$

   $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

2. *Condition for the given line to be parallel to the given plane*

   The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is parallel to the plane

   $a_2x + b_2y + c_2z + d = 0$

   $\Leftrightarrow$ the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is perpendicular to the normal to the plane $a_2x + b_2y + c_2z + d = 0$

   $\Leftrightarrow$ $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

3. *Distance between a line and a plane, parallel to each other*

   If the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ then the distance between them is $\frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}$, which is the same as the distance of a point from the plane.

4. *Length of perpendicular from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz = d$ is*

   $$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

SUMMARY

1. If ϕ is the angle between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \vec{n} = q$ then

$$\sin \phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}.$$

2. The line $\vec{r} = \vec{a} + \lambda \vec{b}$ is **perpendicular** to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} = t\vec{n}$ for some scalar t.

3. (i) The line $\vec{r} = \vec{a} + \lambda \vec{b}$ is **parallel** to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} \cdot \vec{n} = 0$.

   (ii) If the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ then the distance between them is

$$\frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|}.$$

4. If ϕ is the angle between the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z + d = 0$ then

$$\sin \phi = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

5. The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is **perpendicular** to the plane $a_2x + b_2y + c_2z + d = 0$ only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

6. The line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is **parallel** to the plane $a_2x + b_2y + c_2z + d = 0$ only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the angle between the line* $\vec{r} = (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + \hat{k})$ *and the plane* $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 5$.

SOLUTION We know that the angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the

plane $\vec{r} \cdot \vec{n} = q$ is given by

$$\sin\phi = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}.$$

Here, $\vec{b} = (2\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{n} = (6\hat{i} - 3\hat{j} + 2\hat{k})$.

$$\therefore \quad \sin\phi = \frac{|(2\hat{i} + 2\hat{j} + \hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k})|}{\left\{\sqrt{2^2 + 2^2 + 1^2}\right\}\left\{\sqrt{6^2 + (-3)^2 + 2^2}\right\}}$$

$$= \frac{|(12 - 6 + 2)|}{\{\sqrt{9} \times \sqrt{49}\}} = \frac{8}{3 \times 7} = \frac{8}{21}$$

$$\Rightarrow \quad \phi = \sin^{-1}\left(\frac{8}{21}\right).$$

Hence, the angle between the given line and the given plane is $\sin^{-1}\left(\frac{8}{21}\right)$.

EXAMPLE 2 *Find the value of m for which the line* $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ *is parallel to the plane* $\vec{r} \cdot (3\hat{i} - 2\hat{j} + m\hat{k}) = 12$.

SOLUTION We know that the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} \cdot \vec{n} = 0$.

Here, $\vec{b} = (2\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{n} = (3\hat{i} - 2\hat{j} + m\hat{k})$.

So, the given line is parallel to the given plane

$$\Leftrightarrow \quad \vec{b} \cdot \vec{n} = 0$$

$$\Leftrightarrow \quad (2\hat{i} + \hat{j} + 2\hat{k}) \cdot (3\hat{i} - 2\hat{j} + m\hat{k}) = 0$$

$$\Leftrightarrow \quad (2 \times 3) + 1 \times (-2) + 2 \times m = 0$$

$$\Leftrightarrow \quad 2m = -4 \Leftrightarrow m = -2.$$

Hence, the required value of m is –2.

EXAMPLE 3 *Show that the line* $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ *is parallel to the plane* $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$.

Also, find the distance between the given line and the given plane.

[CBSE 2010]

SOLUTION The given line is in the form $\vec{r} = \vec{a} + \lambda \vec{b}$ and the given plane is in the form $\vec{r} \cdot \vec{n} = q$, where

$\vec{a} = (2\hat{i} - 2\hat{j} + 3\hat{k})$, $\vec{b} = (\hat{i} - \hat{j} + 4\hat{k})$, $\vec{n} = (\hat{i} + 5\hat{j} + \hat{k})$ and $q = 5$.

We know that the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = q$ only when $\vec{b} \cdot \vec{n} = 0$.

Here, $\vec{b} \cdot \vec{n} = (\hat{i} - \hat{j} + 4\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k})$

$= (1 \times 1) + (-1) \times 5 + 4 \times 1 = 0.$

Hence, the given line is parallel to the given plane.

Distance between the given line and the given plane

$$= \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|} = \frac{|(2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) - 5|}{|\hat{i} + 5\hat{j} + \hat{k}|}$$

$$= \frac{|(2 \times 1) + (-2) \times 5 + (3 \times 1) - 5|}{\sqrt{1^2 + 5^2 + 1^2}}$$

$$= \frac{|2 - 10 + 3 - 5|}{\sqrt{27}} = \frac{10}{3\sqrt{3}} \text{ units.}$$

Hence, the distance between the given line and the given plane is $\frac{10}{3\sqrt{3}}$ units.

EXAMPLE 4 *Find the vector equation of a line passing through the point* $A(1, -1, 2)$ *and perpendicular to the plane* $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 5$.

SOLUTION The position vector of the given point is $\vec{a} = (\hat{i} - \hat{j} + 2\hat{k})$.

The given plane is $\vec{r} \cdot \vec{n} = q$; where $\vec{n} = (2\hat{i} - \hat{j} + 3\hat{k})$ and $q = 5$.

Since the required line is perpendicular to the given plane, so it must be parallel to its normal $\vec{n}$.

Thus, we need a line which passes through a point having position vector $\vec{a}$ and which is parallel to vector $\vec{n}$.

So, its vector equation is

$$\vec{r} = \vec{a} + \lambda \vec{n}, \text{ i.e., } \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda (2\hat{i} - \hat{j} + 3\hat{k})$$

for some scalar λ.

EXAMPLE 5 *Find the angle between the line* $\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z+4}{3}$ *and the plane* $2x-3y+z=5$.

SOLUTION The given line is $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$, where $a_1=-1, b_1=2, c_1=3$.

The given plane is $a_2x+b_2y+c_2z+d=0$, where $a_2=2$, $b_2=-3$, $c_2=1, d=-5$.

Let the required angle be ϕ. Then

$$\sin\phi=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\left\{\sqrt{a_1^2+b_1^2+c_1^2}\right\}\cdot\left\{\sqrt{a_2^2+b_2^2+c_2^2}\right\}}$$

$$=\frac{|(-1)\times2+2\times(-3)+3\times1|}{\left\{\sqrt{(-1)^2+2^2+3^2}\right\}\cdot\left\{\sqrt{2^2+(-3)^2+1^2}\right\}}$$

$$=\frac{|-2-6+3|}{\{\sqrt{14}\times\sqrt{14}\}}=\frac{|-5|}{14}=\frac{5}{14}$$

$$\Rightarrow\quad \phi=\sin^{-1}\left(\frac{5}{14}\right).$$

Hence, the angle between the given line and and the given plane is $\sin^{-1}\left(\frac{5}{14}\right)$.

EXAMPLE 6 *Find the equation of the line passing through the point* $(3, 0, 1)$ *and parallel to the planes* $x+2y=0$ *and* $3y-z=0$. **[CBSE 2012]**

SOLUTION Let the direction ratios of the required line be a, b, c.

Then, its equation is given by

$$\frac{x-3}{a}=\frac{y-0}{b}=\frac{z-1}{c}. \qquad \text{... (i)}$$

Since the line (i) is parallel to each of the planes $x+2y+0z=0$ and $0x+3y-z=0$, so it must be perpendicular to the normal of each of these planes.

$\therefore\quad a\times1+b\times2+c\times0=0\ \Rightarrow\ a+2b+0c=0$ … (ii)

and $a\times0+b\times3+c\times(-1)=0\ \Rightarrow\ 0a+3b-c=0$ … (iii)

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(-2-0)}=\frac{b}{(0+1)}=\frac{c}{(3-0)}$$

$$\Rightarrow\quad \frac{a}{-2}=\frac{b}{1}=\frac{c}{3}=\lambda\ (\text{say})\ \Rightarrow\ a=-2\lambda,\ b=\lambda \text{ and } c=3\lambda.$$

Putting these values of a, b, c in (i), we get

$$\frac{x-3}{-2\lambda}=\frac{y-0}{\lambda}=\frac{z-1}{3\lambda} \Rightarrow \frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}.$$

Hence, the required equation of the line is $\frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}$.

EXAMPLE 7 *Find the equation of the plane passing through the point $A(1, 2, 1)$ and perpendicular to the line joining the points $P(1, 4, 2)$ and $Q(2, 3, 5)$. Also find the distance of this plane from the line* $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$. **[CBSE 2010C, '11]**

SOLUTION The general equation of a plane passing through the point $A(1, 2, 1)$ is given by

$$a(x-1)+b(y-2)+c(z-1)=0. \quad \text{... (i)}$$

D.r.'s of noraml to the plane (i) are a, b, c.

D.r.'s of line PQ are $(2-1), (3-4), (5-2)$, i.e., $1, -1, 3$.

Since the required plane is perpendicular to line PQ, so the normal to this plane must be parallel to PQ.

$$\therefore \quad \frac{a}{1}=\frac{b}{-1}=\frac{c}{3}=\lambda \text{ (say)} \Rightarrow a=\lambda,\ b=-\lambda \text{ and } c=3\lambda.$$

Putting these values in (i), we get the required equation of the plane as $\lambda(x-1)-\lambda(y-2)+3\lambda(z-1)=0$

$$\Rightarrow \quad (x-1)-(y-2)+3(z-1)=0 \Rightarrow x-y+3z=2 \quad \text{... (ii)}$$

Hence, the required equation of the plane is $x-y+3z=2$.

The given line is $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$. ... (iii)

This line passes through the point $(-3, 5, 7)$.

Distance between the plane (ii) and the line (iii)

= Distance of any point on this line from the plane

= Length of perpendicular from $(-3, 5, 7)$ on plane (ii)

$$=\frac{|-3-5+21-2|}{\sqrt{1^2+(-1)^2+3^2}}=\frac{11}{\sqrt{11}}=\sqrt{11} \text{ units.}$$

Hence, the distance between the desired plane and the given line is $\sqrt{11}$ units.

EXAMPLE 8 *Find the equation of the plane passing through the points $(0, 0, 0)$ and $(3, -1, 2)$ and parallel to the line* $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$. **[CBSE 2011]**

SOLUTION The general equation of the plane passing through the point $(3, -1, 2)$ is given by

$$a(x-3)+b(y+1)+c(z-2)=0. \quad \text{... (i)}$$

If this plane passes through the point $(0, 0, 0)$, we have

$$a(0-3)+b(0+1)+c(0-2)=0 \Rightarrow -3a+b-2c=0. \quad \text{... (ii)}$$

D.r.'s of the normal to the plane (i) are a, b, c.

D.r.'s of the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ are $1, -4, 7$.

The required plane is parallel to the given line only when the normal to this plane is perpendicular to this line.

$\therefore \quad (a \times 1) + b \times (-4) + c \times 7 = 0 \Rightarrow a - 4b + 7c = 0.$ … (iii)

On solving (ii) and (iii) by cross multiplication, we get:

$$\frac{a}{(7-8)} = \frac{b}{(-2+21)} = \frac{c}{(12-1)}$$

$$\Rightarrow \quad \frac{a}{-1} = \frac{b}{19} = \frac{c}{11} = \lambda (\text{say}) \Rightarrow a = -\lambda,\ b = 19\lambda \text{ and } c = 11\lambda.$$

Putting these values of a, b, c in (i), we get:

$$-\lambda(x-3) + 19\lambda(y+1) + 11\lambda(z-2) = 0$$

$$\Rightarrow \quad (x-3) - 19(y+1) - 11(z-2) = 0$$

$$\Rightarrow \quad x - 19y - 11z = 0.$$

Hence, the required equation of the plane is $x - 19y - 11z = 0$.

EXAMPLE 9 *Find the equation of the plane which passes through the point* $(4, -1, 2)$ *and which is parallel to each of the lines* $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z-4}{3}$ *and* $\frac{x+2}{3} = \frac{y-2}{-1} = \frac{z+1}{2}$.

SOLUTION The general equation of a plane passing through the point $A(4, -1, 2)$ is given by

$$a(x-4) + b(y+1) + c(z-2) = 0 \quad \text{... (i)}$$

This plane will be parallel to each of the given lines only when the normal to the plane is perpendicular to each of the given lines.

$\therefore \quad (1 \times a) + (2 \times b) + (3 \times c) = 0 \Rightarrow a + 2b + 3c = 0.$ ... (ii)

$(3 \times a) + (-1) \times b + (2 \times c) = 0 \Rightarrow 3a - b + 2c = 0$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(4+3)} = \frac{b}{(9-2)} = \frac{c}{(-1-6)}$$

$$\Rightarrow \quad \frac{a}{7} = \frac{b}{7} = \frac{c}{-7} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{-1} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = \lambda,\ b = \lambda \text{ and } c = -\lambda.$$

Putting these values of a, b, c in (i), we get

$$\lambda(x-4) + \lambda(y+1) - \lambda(z-2) = 0$$

$$\Rightarrow \quad (x-4) + (y+1) - (z-2) = 0 \Rightarrow x + y - z = 1.$$

Hence, the required equation of the plane is $x + y - z = 1$.

EXAMPLE 10 *Find the equation of the plane passing through* $(2, 3, -4)$ *and* $(1, -1, 3)$ *and parallel to the x-axis.*

SOLUTION The general equation of the plane passing through the point $A(2, 3, -4)$ is given by

$$a(x-2)+b(y-3)+c(z+4)=0. \quad \text{... (i)}$$

Since it passes through the point $B(1, -1, 3)$, we have

$$a(1-2)+b(-1-3)+c(3+4)=0 \Rightarrow -a-4b+7c=0. \quad \text{... (ii)}$$

If this plane is parallel to x-axis, then the normal to the plane is perpendicular to the x-axis.

D.r.'s of normal to plane (i) are a, b, c and d.r.'s of the x-axis are 1, 0, 0.

$\therefore \quad a\times 1+b\times 0+c\times 0=0 \Rightarrow a=0.$

Putting $a=0$ in (ii), we get $7c-4b=0$.

Let $b=\lambda$. Then, $7c=4\lambda \Rightarrow c=\frac{4\lambda}{7}$.

Putting $a=0$, $b=\lambda$ and $c=\frac{4\lambda}{7}$ in (i), we get

$$0\times(x-2)+\lambda(y-3)+\frac{4\lambda}{7}(z+4)=0$$

$$\Rightarrow \quad (y-3)+\frac{4}{7}(z+4)=0 \Rightarrow 7y-21+4z+16=0$$

$$\Rightarrow \quad 7y+4z-5=0.$$

Hence, the required equation of the plane is $7y+4z-5=0$.

EXAMPLE 11 *Find the equation of the plane passing through the point* $(0, 7, -7)$ *and containing the line* $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

SOLUTION The general equation of the plane passing through the point $(0, 7, -7)$ is given by

$$a(x-0)+b(y-7)+c(z+7)=0. \quad \text{... (i)}$$

If (i) contains the given line, then it must pass through the point $(-1, 3, -2)$ and must be parallel to the given line.

If (i) passes through the point $(-1, 3, -2)$, we have

$$a(-1-0)+b(3-7)+c(-2+7)=0 \Rightarrow a+4b-5c=0. \quad \text{... (ii)}$$

If (i) is parallel to the given line, then its normal should be perpendicular to this line.

$$\therefore \quad (-3)a+2b+1\times c=0 \Rightarrow -3a+2b+c=0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(4+10)}=\frac{b}{(15-1)}=\frac{c}{(2+12)} \Rightarrow \frac{a}{14}=\frac{b}{14}=\frac{c}{14}$$

$$\Rightarrow \quad \frac{a}{1}=\frac{b}{1}=\frac{c}{1}=\lambda \text{ (say)}.$$

Then $a=\lambda$, $b=\lambda$ and $c=\lambda$.

Putting $a = \lambda, b = \lambda$ and $c = \lambda$ in, we get

$$\lambda x + \lambda(y-7) + \lambda(z+7) = 0 \Rightarrow x + (y-7) + (z+7) = 0$$

$$\Rightarrow \quad x + y + z = 0.$$

Hence, the required equation of the plane is $x + y + z = 0$.

EXAMPLE 12 *Find the equation of the plane passing through the line of intersection of the planes* $2x + y - z = 3$ *and* $5x - 3y + 4z + 9 = 0$ *and parallel to the line* $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$. **[CBSE 2011]**

SOLUTION The equation of a plane passing through the intersection of the given planes is given by

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0 \text{ for some real number } \lambda$$

$$\Rightarrow \quad (2+5\lambda)x + (1-3\lambda)y + (4\lambda - 1)z + (9\lambda - 3) = 0 \qquad \text{... (i)}$$

If this plane is parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$, then the normal to the plane is perpendicular to this line.

$$\therefore \quad 2(2+5\lambda) + 4(1-3\lambda) + 5(4\lambda - 1) = 0$$

$$\Rightarrow \quad (10\lambda - 12\lambda + 20\lambda) + (4 + 4 - 5) = 0$$

$$\Rightarrow \quad 18\lambda = -3 \Rightarrow \lambda = \frac{-3}{18} = \frac{-1}{6}.$$

Putting $\lambda = \frac{-1}{6}$ in (i), we get

$$\left(2 - \frac{5}{6}\right)x + \left(1 + \frac{3}{6}\right)y + \left(\frac{-4}{6} - 1\right)z + \left(\frac{-9}{6} - 3\right) = 0$$

$$\Rightarrow \quad 7x + 9y - 10z - 27 = 0.$$

Hence, the required equation of the plane is $7x + 9y - 10z - 27 = 0$.

EXAMPLE 13 *Show that the equation* $by + cz + d = 0$ *represents a plane parallel to the x-axis. Find the equation of a plane which is parallel to the x-axis and passes through the points* $A(2, 3, 1)$ *and* $B(4, -5, 3)$.

SOLUTION The given equation is $0 \cdot x + by + cz + d = 0$, which is of the form $ax + by + cz + d = 0$.

Hence, the given equation represents a plane.

D.r.'s of the normal to this plane are $0, b, c$.

D.r.'s of the x-axis are $1, 0, 0$.

Now $0 \times 1 + b \times 0 + c \times 0 = 0$.

This shows that the given plane is parallel to the x-axis.

Thus, the equation of a plane parallel to the x-axis is

$$by + cz + d = 0. \qquad \text{... (i)}$$

If it passes through the points $A(2, 3, 1)$ and $B(4, -5, 3)$, we have

$$3b + c + d = 0 \qquad \text{... (ii)}$$

$$-5b + 3c + d = 0 \qquad \text{... (ii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{b}{(1-3)} = \frac{c}{(-5-3)} = \frac{d}{(9+5)}$$

$$\Rightarrow \quad \frac{b}{-2} = \frac{c}{-8} = \frac{d}{14} \Rightarrow \frac{b}{1} = \frac{c}{4} = \frac{d}{-7} = k \text{ (say)}$$

$$\Rightarrow \quad b = k,\ c = 4k \text{ and } d = -7k.$$

Putting these values in (i), we get

$$ky + 4kz - 7k = 0 \Rightarrow y + 4z - 7 = 0.$$

Hence, the required equation of the plane is $y + 4z - 7 = 0$.

EXERCISE 28G

1. Find the angle between the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$.
2. Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3$.
3. Find the angle between the line $\vec{r} = (3\hat{i} + \hat{k}) + \lambda(\hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 1$.
4. Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$ and the plane $3x + 4y + z + 5 = 0$.
5. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.
6. Find the angle between the line joining the points $A(3, -4, -2)$ and $B(12, 2, 0)$ and the plane $3x - y + z = 1$.
7. If the plane $2x - 3y - 6z = 13$ makes an angle $\sin^{-1}(\lambda)$ with the x-axis, then find the value of λ.
8. Show that the line $\vec{r} = (2\hat{i} + 5\hat{j} + 7\hat{k}) + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})$ is parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 7$.
   Also, find the distance between them.
9. Find the value of m for which the line $\vec{r} = (\hat{i} + 2\hat{k}) + \lambda(2\hat{i} - m\hat{j} - 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (m\hat{i} + 3\hat{j} + \hat{k}) = 4$.

10. Find the vector equation of a line passing through the origin and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 3$.

11. Find the vector equation of the line passing through the point with position vector $(\hat{i} - 2\hat{j} + 5\hat{k})$ and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} - \hat{k}) = 0$.

12. Show that the equation $ax + by + d = 0$ represents a plane parallel to the z-axis. Hence, find the equation of a plane which is parallel to the z-axis and passes through the points $A(2, -3, 1)$ and $B(-4, 7, 6)$.

13. Find the equation of the plane passing through the points (1, 2, 3) and (0, −1, 0) and parallel to the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-3}$. **[CBSE 2006]**

14. Find the equation of a plane passing through the point (2, −1, 5), perpendicular to the plane $x + 2y - 3z = 7$ and parallel to the line $\frac{x+5}{3} = \frac{y+1}{-1} = \frac{z-2}{1}$.

15. Find the equation of the plane passing through the intersection of the planes $4x - y + z = 10$ and $x + y - z = 4$ and parallel to the line with direction ratios 2, 1, 1. Find also the perpendicular distance of (1, 1, 1) from this plane.

ANSWERS (EXERCISE 28G)

1. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **2.** $\sin^{-1}\left(\frac{4}{\sqrt{42}}\right)$

3. $\sin^{-1}\left(\frac{1}{3\sqrt{2}}\right)$ **4.** $\sin^{-1}\left(\frac{7}{2\sqrt{91}}\right)$

5. $\sin^{-1}\left(\frac{8}{21}\right)$ **6.** $\sin^{-1}\left(\frac{23}{11\sqrt{11}}\right)$

7. $\frac{2}{7}$ **8.** $\frac{7}{\sqrt{3}}$ units

9. $m = -3$ **10.** $\vec{r} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ for some scalar λ

11. $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} - 3\hat{j} - \hat{k})$ for some scalar λ

12. $5x + 3y - 1 = 0$ **13.** $6x - 3y + z = 3$

14. $x + 10y + 7z - 27 = 0$ **15.** $5y - 5z - 6 = 0, \frac{3\sqrt{2}}{5}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28G)

6. Equation of line AB is $\frac{x-3}{(12-3)}=\frac{y+4}{(2+4)}=\frac{z+2}{(0+2)}$, i.e., $\frac{x-3}{9}=\frac{y+4}{6}=\frac{z+2}{2}$.

7. D.r.'s of the x-axis are 1, 0, 0 and d.r.'s of normal to the plane are 2, –3, –6.
Let ϕ be the angle between the x-axis and the given plane. Then,
$$\sin\phi=\frac{|1\times 2+0\times(-3)+0\times(-6)|}{\left\{\sqrt{1^2+0^2+0^2}\right\}\left\{\sqrt{2^2+(-3)^2+(-6)^2}\right\}}=\frac{2}{7}\Rightarrow \phi=\sin^{-1}\left(\frac{2}{7}\right).$$
Hence, $\lambda=\frac{2}{7}$.

8. We know that a line $\vec{r}=\vec{a}+\lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n}=q$ only when this line is perpendicular to the normal to the plane.
So, we must have $\vec{b}\cdot\vec{n}=0$.
Here, $(\vec{b}\cdot\vec{n})=(\hat{i}+3\hat{j}+4\hat{k})\cdot(\hat{i}+\hat{j}-\hat{k})=(1+3-4)=0$.
Hence, the given line is parallel to the given plane.
Required distance between the line and the plane
$$=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}=\frac{|(2\hat{i}+5\hat{j}+7\hat{k})\cdot(\hat{i}+\hat{j}-\hat{k})-7|}{|\hat{i}+\hat{j}-\hat{k}|}$$
$$=\frac{|2+5-7-7|}{\sqrt{1^2+1^2+(-1)^2}}=\frac{|-7|}{\sqrt{3}}=\frac{7}{\sqrt{3}}\text{ units.}$$

12. The given equation is $ax+by+0\cdot z+d=0$ which is of the form $ax+by+cz+d=0$. Therefore, it represents a plane.
D.r.'s of normal to the plane are a, b, 0.
D.r.'s of the z-axis are 0, 0, 1.
Now, $a\times 0+b\times 0+0\times 1=0$.
This shows that the given plane is parallel to the z-axis.
Let the required plane be $ax+by+d=0$. ... (i)
Since it passes through the points $A(2,-3,1)$ and $B(-4,7,6)$, we have
$2a-3b+d=0$ ... (ii)
$-4a+7b+d=0$ ... (iii)
On solving (ii) and (iii) by cross multiplication, we get
$$\frac{a}{(-3-7)}=\frac{b}{(-4-2)}=\frac{c}{(14-12)}$$
$$\Rightarrow\ \frac{a}{-10}=\frac{b}{-6}=\frac{c}{2}\Rightarrow\frac{a}{5}=\frac{b}{3}=\frac{c}{-1}=k\text{ (say).}$$
$\therefore\ a=5k$, $b=3k$ and $c=-k$.
Putting these values in (i), we get
$5kx+3ky-k=0\Rightarrow 5x+3y-1=0$,
which is the required equation of the plane.

13. Any plane through the point (1, 2, 3) is

$$a(x-1)+b(y-2)+c(z-3)=0. \quad \text{... (i)}$$

Since it passes through the point (0, –1, 0), we have

$$a(0-1)+b(-1-2)+c(0-3)=0 \Rightarrow a+3b+3c=0 \quad \text{... (ii)}$$

This plane is parallel to the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z}{-3}$, so

$$2a+3b-3c=0 \quad \text{... (iii)}$$

On solving (ii) and (iii), we get

$$\frac{a}{(9+9)}=\frac{b}{(-3-6)}=\frac{c}{(6-3)} \Rightarrow \frac{a}{18}=\frac{b}{-9}=\frac{c}{3} \Rightarrow \frac{a}{6}=\frac{b}{-3}=\frac{c}{1}=\lambda \text{ (say).}$$

Hence, the required equation of the plane is

$$6\lambda(x-1)-3\lambda(y-2)+\lambda(z-3)=0 \Rightarrow 6x-3y+z=3.$$

Equation of a Plane Passing through a Given Point and Parallel to Two Given Lines

Vector Form

THEOREM 1 *The vector equation of a plane passing through a given point with position vector* $\vec{a}$ *and parallel to two given vectors* $\vec{b}$ *and* $\vec{c}$ *is* $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0.$

PROOF The required plane is parallel to vectors $\vec{b}$ and $\vec{c}$.

So, the vector $\vec{n}=(\vec{b}\times\vec{c})$ is perpendicular to this plane.

Thus, we have to find the equation of a plane passing through the point with position vector $\vec{a}$ and perpendicular to the vector $\vec{n}$.

So, its equation is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0, \text{ i.e., } (\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0.$$

Hence, the required equation of the plane is $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0.$

Cartesian Form

THEOREM 2 *The equation of the plane passing through a given point* $A(x_1, y_1, z_1)$ *and parallel to two given lines having direction ratios* b_1, b_2, b_3 *and* c_1, c_2, c_3 *is*

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

PROOF Let us consider a plane passing through a given point $A(x_1, y_1, z_1)$ and parallel to two given lines having direction ratios b_1, b_2, b_3 and c_1, c_2, c_3.

Let $P(x, y, z)$ be an arbitrary point on the plane. Then,

$$\overrightarrow{AP} = (\text{p.v. of } P) - (\text{p.v. of } A)$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}.$$

It is given that the plane is parallel to two lines having direction ratios b_1, b_2, b_3 and c_1, c_2, c_3.

So, the given plane is parallel to each of the vectors

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

$\therefore \overrightarrow{AP}$, $\vec{b}$ and $\vec{c}$ are coplanar, and therefore their scalar triple product must be zero.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0,$$

which is the required equation.

SUMMARY OF THE RESULTS

(i) The vector equation of a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$, is given by

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

(ii) The Cartesian equation of a plane passing through a point $A(x_1, y_1, z_1)$ and parallel to two nonparallel lines having d.r.'s b_1, b_2, b_3 and c_1, c_2, c_3 is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector and Cartesian equations of the plane passing through the point* $(1, 2, -4)$ *and parallel to the lines*

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) - \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $\vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$.

SOLUTION We know that the vector equation of a plane passing through a point $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is given by $(\vec{r} \cdot \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$.

Here $\vec{a} = (\hat{i} + 2\hat{j} - 4\hat{k})$, $\vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k})$

and $\vec{c} = (\hat{i} + \hat{j} - \hat{k})$.

$$\therefore \quad \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (-3-6)\hat{i} - (-2-6)\hat{j} + (2-3)\hat{k} = (-9\hat{i} + 8\hat{j} - \hat{k}).$$

So, the required vector equation is

$$\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\Rightarrow \quad \vec{r} \cdot (-9\hat{i} + 8\hat{j} - \hat{k}) = (\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (-9\hat{i} + 8\hat{j} - \hat{k})$$

$$= (-9 + 16 + 4) = 11$$

$$\Rightarrow \quad \vec{r} \cdot (-9\hat{i} + 8\hat{j} - \hat{k}) - 11 = 0$$

$$\Rightarrow \quad \vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0.$$

Hence, the required vector equation of the plane is

$$\vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0.$$

Clearly, the required Cartesian equation of the plane is

$$9x - 8y + z + 11 = 0.$$

EXAMPLE 2 *Find the Cartesian and vector equation of the plane passing through the point* $(2, 0, -1)$ *and parallel to the lines*

$$\frac{x}{-3} = \frac{y-2}{4} = z + 1 \text{ and } x - 4 = \frac{1-y}{2} = 2z.$$

SOLUTION The given lines are

$$\frac{x}{-3} = \frac{y-2}{4} = \frac{z+1}{1} \text{ and } \frac{x-4}{1} = \frac{y-1}{-2} = \frac{z}{\frac{1}{2}},$$

i.e., $\frac{x}{-3} = \frac{y-2}{4} = \frac{z+1}{1}$ and $\frac{x-4}{2} = \frac{y-1}{-4} = \frac{z}{1}$.

So, the required plane passes through the point $A(2, 0, -1)$ and it is parallel to the lines having direction ratios $-3, 4, 1$ and $2, -4, 1$.

Here $(x_1 = 2, y_1 = 0, z_1 = -1), (a_1 = -3, b_1 = 4, c_1 = 1)$

and $(a_2 = 2, b_2 = -4, c_2 = 1)$.

The required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-0 & z+1 \\ -3 & 4 & 1 \\ 2 & -4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(4+4) - y(-3-2) + (z+1)(12-8) = 0$$

$$\Rightarrow 8(x-2) + 5y + 4(z+1) = 0$$

$$\Rightarrow 8x + 5y + 4z - 12 = 0.$$

Hence, the required Cartesian equation of the plane is $8x + 5y + 4z = 12$ and its corresponding vector equation is $\vec{r} \cdot (8\hat{i} + 5\hat{j} + 4\hat{k}) = 12$.

EXERCISE 28H

1. Find the vector and Cartesian equations of the plane passing through the origin and parallel to the vectors $(\hat{i} + \hat{j} - \hat{k})$ and $(3\hat{i} - \hat{k})$.
2. Find the vector and Cartesian equations of the plane passing through the point (3, –1, 2) and parallel to the lines $\vec{r} = (-\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 5\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i} - 3\hat{j} + \hat{k}) + \mu(-5\hat{i} + 4\hat{j})$.
3. Find the vector equation of a plane passing through the point (1, 2, 3) and parallel to the lines whose direction ratios are 1, –1, –2, and –1, 0, 2.
4. Find the Cartesian and vector equations of a plane passing through the point (1, 2, –4) and parallel to the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+1}{6}$ and $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z}{-1}$.
5. Find the vector equation of the plane passing through the point $(3\hat{i} + 4\hat{j} + 2\hat{k})$ and parallel to the vectors $(\hat{i} + 2\hat{j} + 3\hat{k})$ and $(\hat{i} - \hat{j} + \hat{k})$.

ANSWERS (EXERCISE 28H)

1. $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0,\ x + 2y + 3z = 0$

2. $\vec{r} \cdot (4\hat{i} + 5\hat{j} - 17\hat{k}) + 27 = 0,\ 4x + 5y - 17z + 27 = 0$

3. $\vec{r} \cdot (2\hat{i} + \hat{k}) = 5$ **4.** $9x - 8y + z + 11 = 0$, $\vec{r} \cdot (9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0$

5. $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

Condition for the Coplanarity of Two Lines

Vector Form

THEOREM 1 *The condition for two lines* $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ *and* $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ *to be coplanar is that*

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0, \text{ i.e., } [\vec{a_2} - \vec{a_1} \quad \vec{b_1} \quad \vec{b_2}] = 0.$$

Also, the equation of the plane containing both these lines is

$$(\vec{r} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0 \quad \text{or} \quad (\vec{r} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$$

PROOF The equations of the given lines are

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \qquad \ldots \text{(i)}$$

$$\vec{r} = \vec{a_2} + \mu \vec{b_2}. \qquad \ldots \text{(ii)}$$

The line (i) passes through a point A with position vector $\vec{a_1}$ and is parallel to $\vec{b_1}$.

The line (ii) passes through a point B with position vector $\vec{a_2}$ and is parallel to $\vec{b_2}$.

Now, $\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A) = (\vec{a_2} - \vec{a_1})$.

$\therefore$ the given lines are coplanar

$\Leftrightarrow$ $\overrightarrow{AB}$, $\vec{b_1}$, $\vec{b_2}$ are coplanar

$\Leftrightarrow$ $\overrightarrow{AB} \cdot (\vec{b_1} \times \vec{b_2}) = 0$

$\Leftrightarrow$ $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$

Equation of the Plane Containing Both the Lines

If the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ are coplanar then their common plane is perpendicular to the vector $(\vec{b_1} \times \vec{b_2})$, and this plane passes through each of the points $\vec{a_1}$ and $\vec{a_2}$.

So, its equation is

$$(\vec{r} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0, \text{ or } (\vec{r} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$$

Cartesian Form

THEOREM 2 *The lines*

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad \textit{and} \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

are coplanar $\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

And, the equation of the plane containing both these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

PROOF The given lines are

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \quad \ldots \text{(i)}$$

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}. \quad \ldots \text{(ii)}$$

The line (i) passes through a point $A(x_1, y_1, z_1)$ and is parallel to the vector $\vec{u_1} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$.

Also, the line (ii) passes through a point $B(x_2, y_2, z_2)$ and is parallel to the vector $\vec{u_2} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$.

Now, $\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$$= (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})$$

$$= (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}.$$

$\therefore$ lines (i) and (ii) are coplanar

$\Leftrightarrow$ there is a plane which passes through the points A and B, and which is parallel to each of $\vec{u_1}$ and $\vec{u_2}$

$\Leftrightarrow$ $\overrightarrow{AB}, \vec{u_1}, \vec{u_2}$ are parallel to the same plane

$\Leftrightarrow$ $\overrightarrow{AB}, \vec{u_1}, \vec{u_2}$ are coplanar

$\Leftrightarrow$ $[\overrightarrow{AB}, \vec{u_1}, \vec{u_2}] = 0$

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Equation of the Plane Containing Both the Lines

If the lines (i) and (ii) are coplanar then their common plane is the plane containing the line (i) and parallel to the line (ii) or it is the plane containing the line (ii) and parallel to the line (i).

Hence, its equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

SUMMARY

1. (i) Two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$ are coplanar only when $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$

   (ii) If two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$ are coplanar, then the equation of the plane containing both of these lines is given by

   $$\{(\vec{r} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0\} \text{ or } \{(\vec{r} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2}) = 0\}.$$

2. (i) The lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar only when $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

   (ii) The equation of the common plane is

   $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Show that the lines*

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \text{ and } \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

are coplanar. Also, find the equation of the plane containing both these lines. **[CBSE 2013C]**

SOLUTION We know that the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ are coplanar

$$\Leftrightarrow (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0.$$

Here, $\vec{a_1} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b_1} = 3\hat{i} - \hat{j}$, $\vec{a_2} = 4\hat{i} - \hat{k}$ and $\vec{b_2} = 2\hat{i} + 3\hat{k}$.

$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (4\hat{i} - \hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = (3\hat{i} - \hat{j}).$$

$$(\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= (-3-0)\hat{i} - (9-0)\hat{j} + (0+2)\hat{k} = -3\hat{i} - 9\hat{j} + 2\hat{k}.$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (3\hat{i} - \hat{j}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k})$$

$$= [3 \times (-3) + (-1) \times (-9) + 0 \times 2] = 0.$$

Hence, the given lines are coplanar.

The equation of the plane containing both the given lines is given by

$$(\vec{r} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0 \Leftrightarrow \vec{r} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_1} \cdot (\vec{b_1} \times \vec{b_2})$$

$$\Leftrightarrow \vec{r} \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = (\hat{i} + \hat{j} - \hat{k}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k})$$

$$\Leftrightarrow \vec{r} \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = (-3 - 9 - 2) \Leftrightarrow \vec{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14.$$

Hence, the required equation of the plane is

$$\vec{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14.$$

EXAMPLE 2 *Show that the lines* $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ *and* $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ *are coplanar. Also, find the equation of the plane containing these lines.*

[CBSE 2013C]

SOLUTION We know that the lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}, \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are coplanar only when $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$ and $x_2 = -1$, $y_2 = 2$, $z_2 = 5$.

$a_1 = -3$, $b_1 = 1$, $c_1 = 5$ and $a_2 = -1$, $b_2 = 2$, $c_2 = 5$

$$\therefore \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ -1 & 2 & 5 \end{vmatrix} [R_2 \to (R_2 - R_3)]$$

$$= 5(-2+2) = (5 \times 0) = 0 \text{ [expanding by } c_3\text{]}.$$

Hence, the given lines are coplanar.

Equation of the plane containing both these lines is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (x+3)(5-10) - (y-1)(-15+5) + (z-5)(-6+1) = 0$$

$$\Leftrightarrow \quad (-5)(x+3) + 10(y-1) - 5(z-5) = 0$$

$$\Leftrightarrow \quad -5x + 10y - 5z = 0 \Leftrightarrow x - 2y + z = 0.$$

Hence, the required equation plane is $x - 2y + z = 0$.

EXAMPLE 3 *Show that the lines*

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \text{ and } \frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$

are coplanar. Also find the equation of the plane containing them.

SOLUTION The given lines are

$$\frac{x-(a-d)}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-(a+d)}{\alpha+\delta} \quad \ldots \text{(i)}$$

$$\frac{x-(b-c)}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-(b+c)}{\beta+\gamma}. \quad \ldots \text{(ii)}$$

We know that the lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}, \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are coplanar $\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

Here, $x_1 = (a-d),\ y_1 = a,\ z_1 = (a+d);$

$x_2 = (b-c),\ y_2 = b,\ z_2 = (b+c);$

$a_1 = (\alpha-\delta),\ b_1 = \alpha,\ c_1 = (\alpha+\delta);$ and

$a_2 = (\beta-\gamma),\ b_2 = \beta,\ c_2 = (\beta+\gamma).$

$$\therefore \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-c)-(a-d) & b-a & (b+c)-(a+d) \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix}$$

$$= \begin{vmatrix} 2(b-a) & b-a & b+c-a+d \\ 2\alpha & \alpha & \alpha + \delta \\ 2\beta & \beta & \beta + \gamma \end{vmatrix} \quad \{C_1 \to C_1 + C_3\}$$

$= 0$ [$\because$ C_1 and C_2 are proportional].

Hence, the given lines are coplanar.

Equation of the plane containing the given lines is given by

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix} = 0.$$

Applying $C_1 \to C_1 + C_3 - 2C_2$, we get

$$\begin{vmatrix} x - z - 2y & y-a & z-(a+d) \\ 0 & \alpha & \alpha + \delta \\ 0 & \beta & \beta + \gamma \end{vmatrix} = 0$$

$\Rightarrow \quad (x + z - 2y) - [\alpha(\beta + \gamma) - \beta(\alpha + \delta)] = 0$

$\Rightarrow \quad (x + z - 2y)(\alpha\gamma - \beta\delta) = 0 \Rightarrow \quad x + z - 2y = 0.$

Hence, the required equation of the plane is $x + z - 2y = 0$.

PLANE CONTAINING PARALLEL LINES

EXAMPLE 4 *Find the equation of the plane which contains the two parallel lines* $\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$, *and* $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1}$.

SOLUTION Clearly, the plane which contains the two given parallel lines must pass through the points $A(3, -4, 1)$ and $B(-1, 2, 0)$, and it must be parallel to the line having direction ratios 3, 2, 1.

Any plane passing through the point $A(3, -4, 1)$ is given by

$$a(x-3) + b(y+4) + c(z-1) = 0. \quad \ldots \text{(i)}$$

If this plane passes through the point $B(-1, 2, 0)$, we have

$$a(-1-3) + b(2+4) + c(0-1) = 0 \Rightarrow \quad 4a - 6b + c = 0 \quad \ldots \text{(ii)}$$

If the plane (ii) is parallel to the line having direction ratios 3, 2, 1 then

$$3a + 2b + c = 0. \quad \ldots \text{(iii)}$$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{(-6-2)}=\frac{b}{(3-4)}=\frac{c}{(8+18)} \Rightarrow \frac{a}{-8}=\frac{b}{-1}=\frac{c}{26}=\lambda \text{ (say)}$$

$$\Rightarrow a=-8\lambda,\ b=-\lambda \text{ and } c=26\lambda.$$

Putting these values of a, b and c in (i), we get

$$-8\lambda(x-3)-\lambda(y+4)+26\lambda(z-1)=0$$

$$\Rightarrow \quad 8(x-3)+(y+4)-26(z-1)=0 \Rightarrow 8x+y-26z+6=0.$$

Hence, the required equation of the plane is $8x+y-26z+6=0$.

EXERCISE 28I

1. Show that the lines
$\vec{r}=(2\hat{j}-3\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$ and $\vec{r}=(2\hat{i}+6\hat{j}+3\hat{k})+\mu(2\hat{i}+3\hat{j}+4\hat{k})$ are coplanar.
Also find the equation of the plane containing these lines.

2. Find the vector and Cartesian forms of the equations of the plane containing the two lines
$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$
and $\vec{r}=(9\hat{i}+5\hat{j}-\hat{k})+\mu(-2\hat{i}+3\hat{j}+8\hat{k})$.

3. Find the vector and Cartesian equations of a plane containing the two lines
$\vec{r}=(2\hat{i}+\hat{j}-3\hat{k})+\lambda(\hat{i}+2\hat{j}+5\hat{k})$
and $\vec{r}=(3\hat{i}+3\hat{j}+2\hat{k})+\mu(3\hat{i}-2\hat{j}+5\hat{k})$.
Also show that the line $\vec{r}=(2\hat{i}+5\hat{j}+2\hat{k})+p(3\hat{i}-2\hat{j}+5\hat{k})$ lies in the plane. **[CBSE 2011C]**

4. Prove that the lines $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}$ and $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ are coplanar.
Also find the equation of the plane containing these lines.

5. Prove that the lines $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$ and $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ are coplanar. Also find the equation of the plane containing these lines.

6. Show that the lines $\frac{5-x}{-4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{2y-8}{2}=\frac{z-5}{3}$ are coplanar. Find the equation of the plane containing these lines. **[CBSE 2014]**

7. Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ are coplanar.
Find the equation of the plane containing these lines.

8. Show that the lines $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z}{-1}$ and $\frac{x-4}{3}=\frac{y-1}{-2}=\frac{z-1}{-1}$ are coplanar. Also find the equation of the plane containing these lines.

9. Find the equation of the plane which contains two parallel lines given by $\frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}$ and $\frac{x-4}{1}=\frac{y-3}{-4}=\frac{z-2}{5}$.

ANSWERS (EXERCISE 28I)

1. $\vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})+7=0$

2. $\vec{r}\cdot(6\hat{i}-28\hat{j}+12\hat{k})+98=0,\ 6x-28y+12z+98=0$

3. $\vec{r}\cdot(10\hat{i}+5\hat{j}-4\hat{k})=37,\ 10x+5y-4z=37$

4. $x-2y+z+7=0$

5. $x-2y+z=0$

6. $17x-47y-24z+172=0$

7. $x+y+z=0$

8. $2x-5y-16z+13=0$

9. $11x-y-3z=35$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28I)

3. The given lines are $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$, where

$\vec{a_1}=(2\hat{i}+\hat{j}-3\hat{k}),\ \vec{a_2}=(3\hat{i}+3\hat{j}+2\hat{k}),$

$\vec{b_1}=(\hat{i}+2\hat{j}+5\hat{k}),\ \vec{b_2}=(3\hat{i}-2\hat{j}+5\hat{k}).$

$$\therefore\quad (\vec{b_1}\times\vec{b_2})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&5\\3&-2&5\end{vmatrix}=(10+10)\hat{i}-(5-15)\hat{j}+(-2-6)\hat{k}$$

$$=(20\hat{i}+10\hat{j}-8\hat{k}).$$

Vector equation of the required plane is

$$\vec{r}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2})$$

$$\Rightarrow\quad \vec{r}\cdot(20\hat{i}+10\hat{j}-8\hat{k})=(2\hat{i}+\hat{j}-3\hat{k})\cdot(20\hat{i}+10\hat{j}-8\hat{k})$$

$$=(40+10+24)=74$$

$$\Rightarrow\quad \vec{r}\cdot(10\hat{i}+5\hat{j}-4\hat{k})=37. \qquad \text{... (i)}$$

The Cartesian equation is

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(10\hat{i}+5\hat{j}-4\hat{k})=37\ \Rightarrow\ 10x+5y-4z=37. \qquad \text{... (ii)}$$

The third line is $\vec{r}=(2\hat{i}+5\hat{j}+2\hat{k})+p(3\hat{i}-2\hat{j}+5\hat{k}).$ ... (iii)

Now, the line (iii) will lie in the plane (ii) if (2, 5, 2) lies on (ii) and $(3\hat{i} - 2\hat{j} + 5\hat{k})$ is perpendicular to the normal of (ii).

Now, $10 \times 2 + 5 \times 5 - 4 \times 2 = 37$ shows that (2, 5, 2) lies on (ii).

Also $10 \times 3 + 5 \times (-2) - 4 \times 5 = 0$ shows that $(3\hat{i} - 2\hat{j} + 5\hat{k})$ is perpendicular to the normal of (ii).

Hence, the line (iii) lies in plane (ii).

6. The given equations are: $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$.

EXERCISE 28J

Very-Small-Answer Questions

1. Find the direction ratios of the normal to the plane $x + 2y - 3z = 5$.
2. Find the direction cosines of the normal to the plane $2x + 3y - z = 4$.
3. Find the direction cosines of the normal to the plane $y = 3$.
4. Find the direction cosines of the normal to the plane $3x + 4 = 0$.
5. Write the equation of the plane parallel to XY-plane and passing through the point (4, –2, 3).
6. Write the equation of the plane parallel to YZ-plane and passing through the point (–3, 2, 0).
7. Write the general equation of a plane parallel to the x-axis.
8. Write the intercept cut off by the plane $2x + y - z = 5$ on the x-axis. **[CBSE 2011]**
9. Write the intercepts made by the plane $4x - 3y + 2z = 12$ on the coordinate axes.
10. Reduce the equation $2x - 3y + 5z + 4 = 0$ to intercept form and find the intercepts made by it on the coordinate axes.
11. Find the equation of a plane passing through the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
12. Write the value of k for which the planes $2x - 5y + kz = 4$ and $x + 2y - z = 6$ are perpendicular to each other.
13. Find the angle between the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$.
14. Find the angle between the planes $\vec{r} \cdot (\hat{i} + \hat{j}) = 1$ and $\vec{r} \cdot (\hat{i} + \hat{k}) = 3$.
15. Find the angle between the planes $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 5\hat{k}) = 0$ and $\vec{r} \cdot (2\hat{i} - \hat{j} - 2\hat{k}) = 7$.
16. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.

17. Find the angle between the line $\vec{r} = (\hat{i} + \hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$.

18. Find the value of λ such that the line $\frac{x-2}{6} = \frac{y-1}{\lambda} = \frac{z+5}{4}$ is perpendicular to the plane $3x - y - 2z = 7$. **[CBSE 2010C]**

19. Write the equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$. **[CBSE 2014]**

20. Find the length of perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. **[CBSE 2013]**

21. Find the direction cosines of the perpendicular from the origin to the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$.

22. Show that the line $\vec{r} = (4\hat{i} - 7\hat{k}) + \lambda(4\hat{i} - 2\hat{j} + 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (5\hat{i} + 4\hat{j} - 4\hat{k}) = 7$.

23. Find the length of perpendicular from the origin to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$.

24. Find the value of λ for which the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{\lambda}$ is parallel to the plane $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 4$.

25. Write the angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ and the plane $x + y + 4 = 0$.

26. Write the equation of a plane passing through the point (2, –1, 1) and parallel to the plane $3x + 2y - z = 7$.

ANSWERS (EXERCISE 28J)

1. $1, 2, -3$ **2.** $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$ **3.** $0, 1, 0$ **4.** $-1, 0, 0$

5. $z = 3$ **6.** $x = -3$ **7.** $by + cz + d = 0$ **8.** $\frac{5}{2}$

9. $3, -4, 6$ **10.** $-2, \frac{4}{3}, \frac{-4}{5}$ **11.** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **12.** $k = -8$

13. $\cos^{-1}\left(\frac{4}{21}\right)$ **14.** $\frac{\pi}{3}$ **15.** $\frac{\pi}{2}$ **16.** $\sin^{-1}\left(\frac{8}{21}\right)$

17. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **18.** $\lambda = -2$ **19.** $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + a + b + c$

20. 3 units **21.** $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$ **23.** 2 units **24.** $\lambda = \frac{-13}{4}$

25. $\frac{\pi}{4}$ **26.** $3x + 2y - z = 3$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 28J)

1. The direction ratios of the normal to the plane $x + 2y - 3z = 5$ are 1, 2, –3.
2. The given plane is $2x + 3y - z = 4$.
   Direction ratios of the normal to the given plane are 2, 3, –1 and $\sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$.
   Hence, the required direction cosines are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$.
3. Direction ratios of the normal to the plane are 0, 1, 0 and $\sqrt{0^2 + 1^2 + 0^2} = 1$.
   Hence, the required direction cosines are 0, 1, 0.
4. $3x + 4 = 0 \Rightarrow -x = \frac{4}{3}$.
   Direction ratios of the normal to this plane are –1, 0, 0 and $\sqrt{(-1)^2 + 0^2 + 0^2} = 1$.
   Hence, the required direction cosines are –1, 0, 0.
5. Any plane parallel to XY-plane is $z = k$.
   Since it passes through (4, –2, 3), we have $3 = k$.
   Hence, the required equation of the plane is $z = 3$.
6. Any plane parallel to YZ-plane is $x = k$.
   Since it passes through (–3, 2, 0), we have $-3 = k$.
   Hence, the required equation of the plane is $x = -3$.
7. Let the required equation of the plane be $ax + by + cz + d = 0$.
   The d.r.'s of this plane are a, b, c.
   The d.r.'s of the x-axis are 1, 0, 0.
   Normal of the required plane is perpendicular to the x-axis.
   $\therefore \quad (a \times 1) + (b \times 0) + (c \times 0) = 0 \Rightarrow a = 0$.
   Hence, the required equation is $by + cz + d = 0$.
8. $2x + y - z = 5 \Rightarrow \frac{x}{\left(\frac{5}{2}\right)} + \frac{y}{5} + \frac{z}{-5} = 1$.
   $\therefore$ intercept cut off by the given plane on the x-axis is $\frac{5}{2}$.
9. $4x - 3y + 2z = 12 \Rightarrow \frac{4x}{12} + \frac{(-3y)}{12} + \frac{2z}{12} = 1 \Rightarrow \frac{x}{3} + \frac{y}{-4} + \frac{z}{6} = 1$.
   Hence, the required intercepts are 3, –4, 6.
10. The given equation may be written as $-2x + 3y - 5z = 4$

$$\Rightarrow \quad \frac{(-2x)}{4}+\frac{3y}{4}+\frac{(-5z)}{4}=1 \Rightarrow \frac{x}{-2}+\frac{y}{\frac{4}{3}}+\frac{z}{\frac{-4}{5}}=1.$$

$\therefore$ the required intercepts are $-2, \frac{4}{3}, \frac{-4}{5}$.

11. Clearly, the required equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

12. Clearly, the normals of the given planes are perpendicular to each other.

$\therefore \quad (2\times 1)+(-5)\times 2+k\times(-1)=0 \Rightarrow k=(2-10)=-8.$

13. D.r.'s of normals to the given planes are 2, 1, –2 and 3, –6, –2.

$$\therefore \quad \cos\theta=\frac{|(2\times 3)+1\times(-6)+(-2)\times(-2)|}{\left\{\sqrt{2^2+1^2+(-2)^2}\right\}\left\{\sqrt{3^2+(-6)^2+(-2)^2}\right\}}$$

$$=\frac{4}{(\sqrt{9})(\sqrt{49})}=\frac{4}{(3\times 7)}=\frac{4}{21} \Rightarrow \theta=\cos^{-1}\left(\frac{4}{21}\right).$$

14. Given planes are $x+1=1$ and $x+z=3$.

The d.r.'s of normals to these planes are 1, 1, 0 and 1, 0, 1.

$$\therefore \quad \cos\theta=\frac{|(1\times 1)+(1\times 0)+(0\times 1)|}{\left\{\sqrt{1^2+1^2+0^2}\right\}\left\{\sqrt{1^2+0^2+1^2}\right\}}$$

$$=\frac{1}{(\sqrt{2}\times\sqrt{2})}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}.$$

15. The given planes are $3x-4y+5z=0$ and $2x-y-2z=7$.

The d.r.'s of normals to these planes are 3, –4, 5 and 2, –1, –2.

$$\therefore \quad \cos\theta=\frac{|(3\times 2)+(-4)\times(-1)+5\times(-2)|}{\left\{\sqrt{3^2+(-4)^2+5^2}\right\}\left\{\sqrt{2^2+(-1)^2+(-2)^2}\right\}}=0 \Rightarrow \theta=\frac{\pi}{2}.$$

16. D.r.'s of the given line are 2, 3, 6.

D.r.'s of the normal to the given plane are 10, 2, –11.

$$\therefore \quad \sin\theta=\frac{|(2\times 10)+(3\times 2)+6\times(-11)|}{\left\{\sqrt{2^2+3^2+6^2}\right\}\left\{\sqrt{(10)^2+2^2+(-11)^2}\right\}}$$

$$=\frac{40}{\{\sqrt{49}\}\times\{\sqrt{225}\}}=\frac{40}{(7\times 15)}=\frac{8}{21} \Rightarrow \theta=\sin^{-1}\left(\frac{8}{21}\right).$$

17. Given line is $\vec{r}=\vec{a}+\lambda\vec{b}$ and given plane is $\vec{r}\cdot\vec{n}=p$.

$$\sin\theta=\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}=\frac{|(\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})|}{\left\{\sqrt{1^2+(-1)^2+1^2}\right\}\left\{\sqrt{2^2+(-1)^2+1^2}\right\}}$$

$$=\frac{|(2\times 1)+(-1)\times(-1)+1\times 1|}{(\sqrt{3}\times\sqrt{6})}$$

$$= \frac{4}{\sqrt{18}} = \left(\frac{4}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{2\sqrt{2}}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

18. D.r.'s of the given line are 6, λ, -4.

D.r.'s of normal to the given plane are 3, –1, –2.

Given line is parallel to the normal of the plane.

$\therefore \quad \frac{6}{3} = \frac{\lambda}{-1} = \frac{-4}{-2} \Rightarrow \lambda = -2.$

19. Given plane is $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 2 \Rightarrow x + y + z = 2.$

Let the required plane be $x + y + z = k$, where k is a constant.
Since it passes through (a, b, c), we have $k = (a + b + c)$.
So, the required plane is $x + y + z = a + b + c$.
In vector form, it is given by $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c.$

20. We have $p = \frac{|2 \times 0 - 3 \times 0 + 6 \times 0 + 21|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{21}{\sqrt{49}} = \frac{21}{7} = 3$ units.

21. The given equation is $\vec{r} \cdot (-6\hat{i} + 3\hat{j} + 2\hat{k}) = 1.$

D.r.'s of normal to the plane are –6, 3, 2 and $\sqrt{(-6)^2 + 3^2 + 2^2} = \sqrt{49} = 7.$

$\therefore$ d.c.'s of normal to the plane are $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

22. Given line is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{b} = (4\hat{i} - 2\hat{j} + 3\hat{k})$.

D.r.'s of the line are 4, –2, 3.

Given plane is $\vec{r} \cdot \vec{n} = q$, where $\vec{n} = (5\hat{i} + 4\hat{j} - 4\hat{k})$.

D.r.'s of the normal to the given plane are 5, 4, –4.

So, the given line will be parallel to the given plane when this line is perpendicular to the normal to the plane.

Hence, we must have $(4 \times 5) + (-2 \times 4) + 3 \times (-4) = 0$, which is true.

23. We have $\vec{r} \cdot (-2\hat{i} + 3\hat{j} - 6\hat{k}) = 14.$

Here $\vec{n} = (-2\hat{i} + 3\hat{j} - 6\hat{k})$ and $|\vec{n}| = \sqrt{(-2)^2 + 3^2 + (-6)^2} = 7.$

$\therefore \quad \vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{14}{|\vec{n}|} \Rightarrow \vec{r} \cdot \hat{n} = \frac{14}{7} = 2.$

Hence, the length of perpendicular from origin to the given plane is 2 units.

24. Clearly, the given line must be perpendicular to the normal to the given plane.

D.r.'s of the given line are 2, 3, λ.

D.r.'s of the normal to the given plane are 2, 3, 4.

$\therefore \quad (2\times 2)+(3\times 3)+(\lambda\times 4)=0 \Rightarrow 4\lambda=-13 \Rightarrow \lambda=\dfrac{-13}{4}$.

25. D.r.'s of the given line are 2, 1, –2.

D.r.'s of the normal to the given plane are 1, 1, 0.

$$\therefore \quad \sin\theta=\frac{(2\times 1)+(1\times 1)+(-2)\times 0}{\left\{\sqrt{2^2+1^2+(-2)^2}\right\}\left\{\sqrt{1^2+1^2+0^2}\right\}}=\frac{(2+1+0)}{(\sqrt{9})(\sqrt{2})}$$

$$=\frac{3}{3\sqrt{2}}=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}.$$

26. Let the required equation of the plane be $a(x-2)+b(y+1)+c(z-1)=0$.

Here $a=3$, $b=2$ and $c=-1$.

So, the required equation of the plane is

$3(x-2)+2(y+1)-1\cdot(z-1)=0 \Rightarrow 3x+2y-z=3$.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. The direction cosines of the perpendicular from the origin to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+2\hat{k})+1=0$ are

(a) $\dfrac{6}{7},\dfrac{3}{7},\dfrac{-2}{7}$ (b) $\dfrac{6}{7},\dfrac{-3}{7},\dfrac{2}{7}$ (c) $\dfrac{-6}{7},\dfrac{3}{7},\dfrac{2}{7}$ (d) none of these

2. The direction cosines of the normal to the plane $5y+4=0$ are

(a) $0,\dfrac{-4}{5},0$ (b) 0, 1, 0 (c) 0, –1, 0 (d) none of these

3. The length of perpendicular from the origin to the plane $\vec{r}\cdot(3\hat{i}-4\hat{j}-12\hat{k})+39=0$ is

(a) 3 units (b) $\dfrac{13}{5}$ units (c) $\dfrac{5}{3}$ units (d) none of these

4. The equation of a plane passing through the point $A(2,-3,7)$ and making equal intercepts on the axes, is

(a) $x+y+z=3$ (b) $x+y+z=6$ (c) $x+y+z=9$ (d) $x+y+z=4$

5. A plane cuts off intercepts 3, –4, 6 on the coordinate axes. The length of perpendicular from the origin to this plane is

(a) $\dfrac{5}{\sqrt{29}}$ units (b) $\dfrac{8}{\sqrt{29}}$ units (c) $\dfrac{6}{\sqrt{29}}$ units (d) $\dfrac{12}{\sqrt{29}}$ units

6. If the line $\dfrac{x+1}{3}=\dfrac{y-2}{4}=\dfrac{z+6}{5}$ is parallel to the plane $2x-3y+kz=0$, then the value of k is

(a) $\dfrac{5}{6}$ (b) $\dfrac{6}{5}$ (c) $\dfrac{3}{4}$ (d) $\dfrac{4}{5}$

7. If O is the origin and $P(1, 2, -3)$ is a given point, then the equation of the plane through P and perpendicular to OP is

(a) $x + 2y - 3z = 14$ (b) $x - 2y + 3z = 12$

(c) $x - 2y - 3z = 14$ (d) none of these

8. If the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, then the value of k is

(a) –7 (b) 7 (c) 4 (d) –4

9. The plane $2x + 3y + 4z = 12$ meets the coordinate axes in A, B and C. The centroid of $\triangle ABC$ is

(a) (2, 3, 4) (b) (6, 4, 3) (c) $\left(2, \frac{4}{3}, 1\right)$ (d) none of these

10. If a plane meets the coordinate axes in A, B and C such that the centroid of $\triangle ABC$ is (1, 2, 4), then the equation of the plane is

(a) $x + 2y + 4z = 6$ (b) $4x + 2y + z = 12$

(c) $x + 2y + 4z = 7$ (d) $4x + 2y + z = 7$

11. The equation of a plane through the point $A(1, 0, -1)$ and perpendicular to the line $\frac{x+1}{2} = \frac{y+3}{4} = \frac{z+7}{-3}$ is

(a) $2x + 4y - 3z = 3$ (b) $2x - 4y + 3z = 5$

(c) $2x + 4y - 3z = 5$ (d) $x + 3y + 7z = -6$

12. The line $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{4}$ meets the plane $2x + 3y - z = 14$ in the point

(a) (2, 5, 7) (b) (3, 5, 7) (c) (5, 7, 3) (d) (6, 5, 3)

13. The equation of the plane passing through the points $A(2, 2, 1)$ and $B(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 1$, is

(a) $x + 2y - 3z + 5 = 0$ (b) $2x - 3y + 4z - 6 = 0$

(c) $4x + 5y - 6z + 3 = 0$ (d) $3x + 4y - 5z - 9 = 0$

14. The equation of the plane passing through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and passing through the point $A(2, 2, 1)$ is given by

(a) $7x + 5y - 4z - 8 = 0$ (b) $7x - 5y + 4z - 8 = 0$

(c) $5x - 7y + 4z - 8 = 0$ (d) $5x + 7y - 4z + 8 = 0$

15. The equation of the plane passing through the points $A(0, -1, 0)$, $B(2, 1, -1)$ and $C(1, 1, 1)$ is given by

(a) $4x + 3y - 2z - 3 = 0$ (b) $4x - 3y + 2z + 3 = 0$

(c) $4x - 3y + 2z - 3 = 0$ (d) none of these

16. If the plane $2x - y + z = 0$ is parallel to the line $\frac{2x-1}{2} = \frac{2-y}{2} = \frac{z+1}{a}$, then the value of a is

(a) –4 (b) –2 (c) 4 (d) 2

17. The angle between the line $\frac{x+1}{1}=\frac{y}{2}=\frac{z-1}{1}$ and a normal to the plane $x-y+z=0$ is

(a) $0°$ (b) $30°$ (c) $45°$ (d) $90°$

18. The point of intersection of the line $\frac{x-1}{3}=\frac{y+2}{4}=\frac{z-3}{-2}$ and the plane $2x-y+3z-1=0$, is

(a) (–10, 10, 3) (b) (10, 10, –3) (c) (10, –10, 3) (d) (10, –10, –3)

19. The equation of a plane passing through the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ is given by

(a) $ax+by+cz=0$ (b) $ax+by+cz=1$

(c) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ (d) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

20. If θ is the angle between the planes $2x-y+2z=3$ and $6x-2y+3z=5$, then $\cos\theta=?$

(a) $\frac{11}{20}$ (b) $\frac{12}{23}$ (c) $\frac{17}{25}$ (d) $\frac{20}{21}$

21. The angle between the planes $2x-y+z=6$ and $x+y+2z=7$, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

22. The angle between the planes $\vec{r}\cdot(3\hat{i}-6\hat{j}+2\hat{k})=4$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=3$, is

(a) $\cos^{-1}\left(\frac{16}{21}\right)$ (b) $\cos^{-1}\left(\frac{4}{21}\right)$ (c) $\cos^{-1}\left(\frac{3}{4}\right)$ (d) $\cos^{-1}\left(\frac{1}{4}\right)$

23. The equation of the plane through the points $A(2, 3, 1)$ and $B(4, -5, 3)$, parallel to the x-axis, is

(a) $x+y-3z=2$ (b) $y+4z=7$ (c) $y+3z=6$ (d) $x+5y-3z=4$

24. A variable plane moves so that the sum of the reciporcals of its intercepts on the coordinate axes is (1/2). Then, the plane passes through the point

(a) (0, 0, 0) (b) (1, 1, 1) (c) $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ (d) (2, 2, 2)

25. The equation of a plane which is perpendicular to $(2\hat{i}-3\hat{j}+\hat{k})$ and at a distance of 5 units from the origin is

(a) $2x-3y+z=5$ (b) $2x-3y+z=5\sqrt{14}$

(c) $\frac{x}{2}-\frac{y}{3}+\frac{z}{1}=5$ (d) $\frac{x}{2}-\frac{y}{3}+\frac{z}{1}=\frac{5}{\sqrt{14}}$

26. The equation of the plane passing through the point $A(2, 3, 4)$ and parallel to the plane $5x - 6y + 7z = 3$, is

(a) $5x - 6y + 7z = 20$ (b) $7x - 6y + 5z = 72$

(c) $20x - 18y + 14z = 11$ (d) $10x - 18y + 28z = 13$

27. The foot of the perpendicular from the point $A(7, 14, 5)$ to the plane $2x + 4y - z = 2$ is

(a) (3, 1, 8) (b) (1, 2, 8) (c) (3, –3, 5) (d) (5, –3, –4)

28. The equation of the plane which makes with the coordinate axes, a triangle with centroid (α, β, γ) is given by

(a) $\alpha x + \beta y + \gamma z = 1$ (b) $\alpha x + \beta y + \gamma z = 3$

(c) $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$ (d) $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$

29. The intercepts made by the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 12$ are

(a) 2, –3, 4 (b) 2, –3, –6 (c) –6, –4, 3 (d) –6, 4, 3

30. The angle between the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+4}{-3}$ and the plane $2x - 3y + z = 5$ is

(a) $\cos^{-1}\left(\frac{5}{14}\right)$ (b) $\sin^{-1}\left(\frac{5}{14}\right)$ (c) $\cos^{-1}\left(\frac{3}{7}\right)$ (d) $\sin^{-1}\left(\frac{3}{7}\right)$

31. The angle between the line $\vec{r} \cdot (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 5$, is

(a) $\cos^{-1}\left(\frac{8}{21}\right)$ (b) $\cos^{-1}\left(\frac{5}{21}\right)$ (c) $\sin^{-1}\left(\frac{5}{21}\right)$ (d) $\sin^{-1}\left(\frac{8}{21}\right)$

32. The distance of the point $(\hat{i} + 2\hat{j} + 5\hat{k})$ from the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + 17 = 0$, is

(a) $\frac{25}{\sqrt{2}}$ units (b) $\frac{25}{\sqrt{3}}$ units (c) $25\sqrt{2}$ units (d) $25\sqrt{3}$ units

33. The distance between the parallel planes $2x - 3y + 6z = 5$ and $6x - 9y + 18z + 20 = 0$, is

(a) $\frac{5}{3}$ units (b) $5\sqrt{3}$ units (c) $\frac{8}{5}$ units (d) $8\sqrt{5}$ units

34. The distance between the planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$, is

(a) 4 units (b) 2 units (c) $\frac{1}{2}$ units (d) $\frac{1}{4}$ units

35. The image of the point $P(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$, is

(a) (3, –5, 2) (b) (3, 5, –2) (c) (3, 5, 2) (d) (–3, 5, 2)

ANSWERS (OBJECTIVE QUESTIONS)

1. (c) **2.** (c) **3.** (a) **4.** (b) **5.** (d) **6.** (b) **7.** (a) **8** (b) **9.** (c) **10.** (b)
11. (c) **12.** (b) **13.** (d) **14.** (b) **15.** (c) **16.** (a) **17.** (d) **18.** (b) **19.** (d) **20.** (d)
21. (c) **22.** (a) **23.** (b) **24.** (d) **25.** (b) **26.** (a) **27.** (b) **28.** (d) **29.** (c) **30.** (b)
31. (d) **32.** (b) **33.** (a) **34.** (c) **35.** (d)

HINTS TO GIVEN OBJECTIVE QUESTIONS

1. (c) Given plane is $\vec{r}\cdot(-6\hat{i}+3\hat{j}+2\hat{k})=1$.

$\therefore\ \vec{r}\cdot\vec{n}=p$. where $\vec{n}=(-6\hat{i}+3\hat{j}+2\hat{k})$.

D.r.'s of the normal to the given plane are –6, 3, 2 and

$\sqrt{(-6)^2+3^2+2^2}=\sqrt{49}=7$.

$\therefore$ d.c.'s of the normal to the given plane are $\frac{-6}{7},\frac{3}{7},\frac{2}{7}$.

2. (c) $5y+4=0\ \Rightarrow\ -5y=4\ \Rightarrow\ -y=\frac{4}{5}\ \Rightarrow\ 0x-1\cdot y+0\cdot z=\frac{4}{5}$.

D.r.'s of the normal to this plane are 0, –1, 0 and $\sqrt{0^2+(-1)^2+0^2}=\sqrt{1}=1$.

$\therefore$ d.c.'s of the normal to the given plane are 0, –1, 0.

3. (a) Given plane is $\vec{r}\cdot(-3\hat{i}+4\hat{j}+12\hat{k})=39$.

$\vec{r}\cdot\vec{n}=39\ \Rightarrow\ \vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{39}{|\vec{n}|}\Rightarrow\vec{r}\cdot\hat{n}=\frac{39}{|\vec{n}|}$.

And, $|\vec{n}|=\sqrt{(-3)^2+4^2+(12)^2}=\sqrt{169}=13$.

$\therefore\ \vec{r}\cdot\hat{n}=\frac{39}{13}=3$ units. Hence, $p=3$ units.

4. (b) Let the required equation of the plane be $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$, i.e., $x+y+z=a$.

Since, it passes through the point $A(2,-3,7)$, we have $2+(-3)+7=a\ \Rightarrow\ a=6$.
Hence, the required equation of the plane is $x+y+z=6$.

5. (d) The given plane is $\frac{x}{3}+\frac{y}{-4}+\frac{z}{6}=1\ \Rightarrow\ 4x-3y+2z=12$

$\therefore\quad p=\frac{|4\times0-3\times0+2\times0-12|}{\sqrt{4^2+(-3)^2+2^2}}=\frac{12}{\sqrt{29}}$ units.

6. (b) D.r.'s of the given line are 3, 4, 5.

D.r.'s of the normal to the given plane are 2, –3, k.

Since the given line is perpendicular to the normal of the plane, we have

$$(3\times 2)+4\times(-3)+5\times k=0 \Rightarrow 5k=6 \Rightarrow k=\frac{6}{5}.$$

7. (a) Let the required equation of the plane through $P(1, 2, -3)$ be

$$a(x-1)+b(y-2)+c(z+3)=0. \quad \text{... (i)}$$

D.r.'s of OP are $(1-0), (2-0), (-3, 0)$, i.e., $1, 2, -3$.

$\therefore$ $a=1, b=2, c=-3$.

Hence, the required equation of the plane is

$$1(x-1)+2(y-2)-3(z+3)=0 \Rightarrow x+2y-3z=14.$$

8. (b) Clearly, the given line passes through the point $(4, 2, k)$.

Since, the given line lies in the plane $2x-4y+z=7$, so the above point lies in this plane.

$\therefore$ $(2\times 4)-(4\times 2)+k=7 \Rightarrow k=7$.

9. (c) The given equation is $\frac{x}{6}+\frac{y}{4}+\frac{c}{3}=1$.

This plane meets the coordinate axes in $A(6, 0, 0)$, $B(0, 4, 0)$ and $C(0, 0, 3)$.

$\therefore$ centroid of $\triangle ABC$ is $G\left(\frac{6+0+0}{3}, \frac{0+4+0}{3}, \frac{0+0+3}{3}\right)$, i.e., $G\left(2, \frac{4}{3}, 1\right)$.

10. (b) Let the required equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Then, it meets the coordinate axes in $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

$\therefore$ centroid of $\triangle ABC$ is $G\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$, i.e., $G\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

$\therefore$ $\left(\frac{a}{3}=1, \frac{b}{3}=2, \frac{c}{3}=4\right) \Rightarrow a=3, b=6, c=12.$

Hence, the required equation of the plane is

$$\frac{x}{3}+\frac{y}{6}+\frac{z}{12}=1 \Rightarrow 4x+2y+z=12.$$

11. (c) D.r.'s of the given line are $2, 4, -3$.

Let the required equation of plane through $(1, 0, -1)$ be

$$a(x-1)+b(y-0)+c(z+1)=0.$$

Since the line is perpendicular to the plane, so it is parallel to the normal to the plane.

$\therefore$ $a=2, b=4$ and $c=-3$.

Hence, the required equation of the plane is

$$2(x-1)+4(y-0)-3(z+1)=0 \Rightarrow 2x+4y-3z=5.$$

12. (b) Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$ (say).

A general point on this line is $(2\lambda+1, 3\lambda+2, 4\lambda+3)$.

For some value of λ, let the given line meet the plane $2x+3y-z=14$ at a point $P(2\lambda+1, 3\lambda+2, 4\lambda+3)$. Then,

$$2(2\lambda+1)+3(3\lambda+2)-(4\lambda+3)=14 \Rightarrow 9\lambda=9 \Rightarrow \lambda=1.$$

So, the required point is $P(2+1, 3+2, 4+3)$, i.e., $P(3, 5, 7)$.

13. (d) The equation of a plane passing through the point $A(2, 2, 1)$ is

$$a(x-2)+b(y-2)+c(z-1)=0. \quad \text{... (i)}$$

Since it passes through the point $B(9, 3, 6)$, we have

$$a(9-2)+b(3-2)+c(6-1)=0 \Rightarrow 7a+b+5c=0. \quad \text{... (ii)}$$

Also, it being perpendicular to the plane $2x+6y+6z=1$, we have

$$2a+6b+6c=0 \Rightarrow a+3b+3c=0. \quad \text{... (iii)}$$

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{a}{(3-15)}=\frac{b}{(5-21)}=\frac{c}{(21-1)} \Rightarrow \frac{a}{-12}=\frac{b}{-16}=\frac{c}{20} \Rightarrow \frac{a}{3}=\frac{b}{4}=\frac{c}{-5}=k$$

$\Rightarrow$ $a=3k$, $b=4k$ and $c=-5k$

Substituting these values in (i), we get

$$3k(x-2)+4k(y-2)-5k(z-1)=0$$

$$\Rightarrow 3(x-2)+4(y-2)-5(z-1)=0 \Rightarrow 3x+4y-5z-9=0.$$

14. (b) Let the required equation of the plane be

$$(3x-y+2z-4)+\lambda(x+y+z-2)=0$$

$$\Rightarrow (3+\lambda)x+(\lambda-1)y+(2+\lambda)z-(4+2\lambda)=0. \quad \text{... (ii)}$$

Since it passes through the point $A(2, 2, 1)$, we have

$$(3+\lambda)\times 2+(\lambda-1)\times 2+(2+\lambda)\times 1-(4+2\lambda)=0$$

$$\Rightarrow (6+2\lambda)+(2\lambda-2)+(2+\lambda)-(4+2\lambda)=0$$

$$\Rightarrow 3\lambda+2=0 \Rightarrow \lambda=\frac{-2}{3}.$$

So, the required equation of the plane is

$$\left(3-\frac{2}{3}\right)x+\left(\frac{-2}{3}-1\right)y+\left(2-\frac{2}{3}\right)z-\left(4-\frac{4}{3}\right)=0$$

$$\Rightarrow \frac{7x}{3}-\frac{5y}{3}+\frac{4z}{3}-\frac{8}{3}=0 \Rightarrow 7x-5y+4z-8=0.$$

15. (c) Here $(x_1=0, y_1=-1, z_1=0)$, $(x_2=2, y_2=1, z_2=-1)$ and $(x_3=1, y_3=1, z_3=1)$.

So, the required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-0 & y+1 & z-0 \\ 2-0 & 1+1 & -1-0 \\ 1-0 & 1+1 & 1-0 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x & y+1 & z \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow x(2+2)-(y+1)(2+1)+z(4-2)=0$$

$$\Rightarrow 4x-3(y+1)+2z=0 \Rightarrow 4x-3y+2z-3=0.$$

16. (a) The given line is $\frac{x-\frac{1}{2}}{1}=\frac{y-2}{-2}=\frac{z-(-1)}{a}$.

The d.r.'s of the line are $1, -2, a$.

The given plane is $2x-y+z=0$.

The d.r.'s of the normal to this plane are 2, –1, 1.

Since the given plane is parallel to the given line, so the normal to this plane is perpendicular to the given line.

$\therefore \quad (2 \times 1) - 1 \times (-2) + 1 \times a = 0 \Rightarrow a = -4$

17. (d) D.r.'s of the given line are 1, 2, 1.

D.r.'s of the normal to the given plane are 1, –1, 1.

$$\therefore \quad \cos\theta = \frac{|(1 \times 1) + 2 \times (-1) + (1 \times 1)|}{\left\{\sqrt{1^2 + 2^2 + 1^2}\right\}\left\{\sqrt{1^2 + (-1)^2 + 1^2}\right\}} = 0 \Rightarrow \theta = 90°.$$

18. (b) The given line is $\frac{x-1}{3} = \frac{y+2}{4} = \frac{z-3}{-2} = \lambda$ (say).

A general point on this line is $(3\lambda + 1,\ 4\lambda - 2,\ -2\lambda + 3)$.

For some value of λ, let the point $P(3\lambda + 1,\ 4\lambda - 2,\ -2\lambda + 3)$ lie on the plane $2x - y + 3z - 1 = 0$. Then,

$$2(3\lambda + 1) - (4\lambda - 2) + 3(-2\lambda + 3) - 1 = 0$$

$\Rightarrow \quad 4\lambda = 12 \Rightarrow \lambda = 3.$

So, the required point is $P(9 + 1,\ 12 - 2,\ -6 + 3)$, i.e., $P(10,\ 10,\ -3)$.

19. (d) The required equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

20. (d) Angle between the planes means the angle between their normals.

$$\therefore \quad \cos\theta = \frac{|(2 \times 6) + (-1) \times (-2) + 2 \times 3|}{\left\{\sqrt{2^2 + (-1)^2 + 2^2}\right\}\left\{\sqrt{6^2 + (-2)^2 + 3^2}\right\}} = \frac{20}{(\sqrt{9})(\sqrt{49})} = \frac{20}{(3 \times 7)} = \frac{20}{21}.$$

21. (c) Angle between the planes means the angle between their normals.

$$\therefore \quad \cos\theta = \frac{|(2 \times 1) + (-1) \times 1 + (1 \times 2)|}{\left\{\sqrt{2^2 + (-1)^2 + 1^2}\right\}\left\{\sqrt{1^2 + 1^2 + 2^2}\right\}}$$

$$= \frac{3}{(\sqrt{6} \times \sqrt{6})} = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

22. (a) The given planes are $3x - 6y + 2z = 4$ and $2x - y + 2z = 3$.

$$\therefore \quad \cos\theta = \frac{|(3 \times 2) + (-6) \times (-1) + (2 \times 2)|}{\left\{\sqrt{3^2 + (-6)^2 + 2^2}\right\}\left\{\sqrt{2^2 + (-1)^2 + 2^2}\right\}}$$

$$= \frac{16}{(\sqrt{49})(\sqrt{9})} = \frac{16}{(7 \times 3)} = \frac{16}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{16}{21}\right).$$

23. (b) Let the required equation be $by + cz + d = 0$. ... (i)

Since it passes through the points $A(2,\ 3,\ 1)$ and $B(4, -5,\ 3)$, we have

$3b + c + d = 0$ ... (ii)

$-5b + 3c + d = 0.$ ... (iii)

On solving (ii) and (iii) by cross multiplication, we have

$$\frac{b}{(1-3)} = \frac{c}{(-5-3)} = \frac{d}{(9+5)}$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = 5\sqrt{14}.$$

$$\Rightarrow \quad \frac{b}{1} = \frac{c}{4} = \frac{d}{-7} = k \text{ (say)}.$$

Then, $b = k$, $c = 4k$ and $d = -7k$.

Substituting these values in (i), we get the required equation as

$$ky + 4kz - 7k = 0 \Rightarrow y + 4z - 7 = 0 \Rightarrow y + 4z = 7.$$

24. (d) Let the required equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Then, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2} \Rightarrow \frac{2}{a} + \frac{2}{b} + \frac{2}{c} = 1$.

It means that the plane passes through the point (2, 2, 2).

25. (b) We have $|\vec{n}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$.

$$\therefore \quad \vec{r} \cdot \hat{n} = 5 \Rightarrow \vec{r} \cdot \frac{\vec{n}}{\sqrt{14}} = 5$$

$$\Rightarrow \vec{r} \cdot \vec{n} = 5\sqrt{14}.$$

$$\therefore \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 5\sqrt{14} \Rightarrow 2x - 3y + z = 5\sqrt{14}.$$

26. (a) Let the required equation of the plane be $5x - 6y + 7z = k$.
Since it passes through the point $A(2, 3, 4)$, we have

$$(5 \times 2) - 6 \times 3 + 7 \times 4 = k \Rightarrow k = (10 - 18 + 28) = 20.$$

Hence, the required equation of the plane is $5x - 6y + 7z = 20$.

27. (b) Let N be the foot of the perpendicular drawn from the point $A(7, 14, 5)$ and perpendicular to the plane $2x + 4y - z = 2$.

Then, the equation of the line PN is $\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = \lambda$ (say).

Let the coordinates of N be $N(2\lambda + 7, 4\lambda + 14, -\lambda + 5)$.

Since N lies on the plane $2x + 4y - z = 2$, so

$$2(2\lambda + 7) + 4(4\lambda + 14) - (-\lambda + 5) = 2 \Rightarrow 21\lambda = -63 \Rightarrow \lambda = -3.$$

$\therefore$ required foot of the perpendicular is

$$N(-6 + 7, -12 + 14, 3 + 5), \text{ i.e., } N(1, 2, 8).$$

28. (d) Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Then, it meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

$\therefore$ centroid of $\triangle ABC$ is $G\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$, i.e., $G\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

$\therefore \left(\frac{a}{3}=\alpha, \frac{b}{3}=\beta \text{ and } \frac{c}{3}=\gamma\right) \Rightarrow a=3\alpha,\ b=3\beta \text{ and } c=3\gamma.$

$\therefore$ the required equation of the plane is

$$\frac{x}{3\alpha}+\frac{y}{3\beta}+\frac{z}{3\gamma}=1 \Rightarrow \frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$$

29. (c) The given plane is $2x-3y+4z=12 \Rightarrow \frac{x}{6}+\frac{y}{-4}+\frac{z}{3}=1.$

$\therefore$ required intercepts are 6, –4, 3.

30. (b) D.r.'s of the given line are 1, –2, –3.

D.r.'s of the noraml to the given plane are 2, –3, 1.

$$\therefore \sin\theta=\frac{|(1\times 2)+(-2)\times(-3)+(-3)\times 1|}{\left\{\sqrt{1^2+(-2)^2+(-3)^2}\right\}\left\{\sqrt{2^2+(-3)^2+1^2}\right\}}$$

$$=\frac{5}{14} \Rightarrow \theta=\sin^{-1}\left(\frac{5}{14}\right).$$

31. (d) $\sin\theta=\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}.$

Here $\vec{b}=(2\hat{i}+2\hat{j}+\hat{k})$ and $\vec{n}=(6\hat{i}-3\hat{j}+2\hat{k}).$

$$\therefore \sin\theta=\frac{|(2\hat{i}\times 2\hat{j}+\hat{k})\cdot(6\hat{i}-3\hat{j}+2\hat{k})|}{\left\{\sqrt{2^2+2^2+1^2}\right\}\left\{\sqrt{6^2+(-3)^2+2^2}\right\}}$$

$$=\frac{|12-6+2|}{(\sqrt{9})(\sqrt{49})}=\frac{8}{(3\times 7)}=\frac{8}{21} \Rightarrow \theta=\sin^{-1}\left(\frac{8}{21}\right).$$

32. (b) We know that the perpendicular distance of a point having position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n}=q$ is given by

$$p=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}.$$

Here $\vec{a}=(\hat{i}+2\hat{j}+5\hat{k}),\ \vec{n}=(-\hat{i}-\hat{j}-\hat{k})$ and $q=17.$

$$\therefore \quad p=\frac{|(\hat{i}+2\hat{j}+5\hat{k})\cdot(-\hat{i}-\hat{j}-\hat{k})-17|}{|-\hat{i}-\hat{j}-\hat{k}|}$$

$$=\frac{|(-1)+(-2)+(-5)+(-17)|}{\sqrt{(-1)^2+(-1)^2+(-1)^2}}=\frac{25}{\sqrt{3}} \text{ units.}$$

33. (a) The given parallel planes are

$$2x - 3y + 6z = 5 \quad \text{... (i)}$$

$$6x - 9y + 18z + 20 = 0. \quad \text{... (ii)}$$

Let $P(x_1, y_1, z_1)$ be a point on (i). Then, $2x_1 - 3y_1 + 6z_1 = 5$. ... (iii)

Distance between the given planes

= Length of perpendicular from $P(x_1, y_1, z_1)$ on (ii)

$$= \frac{|6x_1 - 9y_1 + 18z_1 + 20|}{\sqrt{6^2 + (-9)^2 + (18)^2}} = \frac{|3(2x_1 - 3y_1 + 6z_1) + 20|}{\sqrt{441}}$$

$$= \frac{|(3 \times 5) + 20|}{21} = \frac{35}{21} \text{ units} = \frac{5}{3} \text{ units} \quad \text{[using (iii)]}$$

34. (c) Clearly the given planes are parallel. These are

$$x + 2y - 2z + 1 = 0 \quad \text{... (i)}$$

$$2x + 4y - 4z + 5 = 0. \quad \text{... (ii)}$$

Let $P(x_1, y_1, z_1)$ be a point on (i). Then, $x_1 + 2y_1 - 2z_1 = -1$.

$$\therefore \quad p = \frac{|2x_1 + 4y_1 - 4z_1 + 5|}{\sqrt{2^2 + 4^2 + (-4)^2}}$$

$$= \frac{|2(x_1 + 2y_1 - 2z_1) + 5|}{\sqrt{36}}$$

$$= \frac{|2 \times (-1) + 5|}{6} = \frac{3}{6} = \frac{1}{2} \text{ units.}$$

35. (d) The equation of a line through the point $P(1, 3, 4)$ and perpendicular to the plane $2x - y + z + 3 = 0$ is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = k \text{ (say)}$$

A general point on this line is $(2k + 1, -k + 3, k + 4)$.

For some value of k let $N(2k + 1, -k + 3, k + 4)$ be a point of the line lying on the plane. Then,

$$2(2k + 1) - (-k + 3) + (k + 4) + 3 = 0 \Rightarrow 6k = -6 \Rightarrow k = -1.$$

$\therefore$ coordinates of N are $N(-2 + 1, 1 + 4, -1 + 4)$, i.e., $N(-1, 4, 3)$.

Let $Q(\alpha, \beta, \gamma)$ be the image of P in the given plane. Then

$$\frac{1+\alpha}{2} = -1, \frac{3+\beta}{2} = 4 \text{ and } \frac{4+\gamma}{2} = 3. \text{ So, } \alpha = -3, \ \beta = 5 \text{ and } \gamma = 2.$$

Hence, the required image of $P(1, 3, 4)$ in the given plane is $Q(-3, 5, 2)$.